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Evolution + entropy + energy = = evo-seti

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Abstract. The discovery of a larger and larger number of Exoplanets raises a question: where does any newly-discovered Exoplanet stand in its capability to develop Life as we know it on Earth? Our answer is our Evo-SETI Theory. This is a mathematical model describing Cladistics and Evolution of Life on Earth by statistical equations based on lognormal probability densities in the time. The lifetime of any living form is a b-lognormal, that is a lognormal probability density starting at time b (= birth). The Shannon Entropy (with reversed sign) of each b-lognormal is the measure of how evolved that Species is. The Molecular Clock is the straight line of the exponential Shannon Entropy enveloping all the b-lognormals. Thermodynamics entails both ENERGY and ENTROPY. It is possible to add ENERGY to ENTROPY by replacing the b-lognormals by a new curve called LOGPAR: an ascending b-lognormal between birth and peak followed by a descending parabola between peak and death. The area under the logpar is the ENERGY needed by that organism to live its entire lifetime. Three instants (birth b, peak p and death d) are sufficient to describe a logpar. As an example, the history of Ancient Rome is a logpar with b=-753, p=117 and d=476. In conclusion, our invention of the logpar led to a new mathematical Evo-SETI Theory describing Evolution, Human History and SETI.

Key words. Biological Evolution – Cladistics – Shannon Information Entropy – Molecular Clock – Energy

1. Introduction

1.1. During the last 3.5 billio years life forms increased like a (lognormal)stochastic process

Let us look at Fig. 1: on the horizontal axis is the time t, with the convention that negative values of t are past times, zero is now, and positive times are future times. The starting point on the time axis is ts (time-of-start) i.e. 3.5 billion years ago, the time of the origin of life on Earth that we assume to be correct. If the origin of life started earlier than that, say 3.8 billion years ago, the coming equations would still be the same and their numerical values will only be slightly changed. On the vertical axis is the number of Species living on Earth at time t, denoted $L(t)$. This "function of the time" we don't know in detail, and so it must be regarded as a random

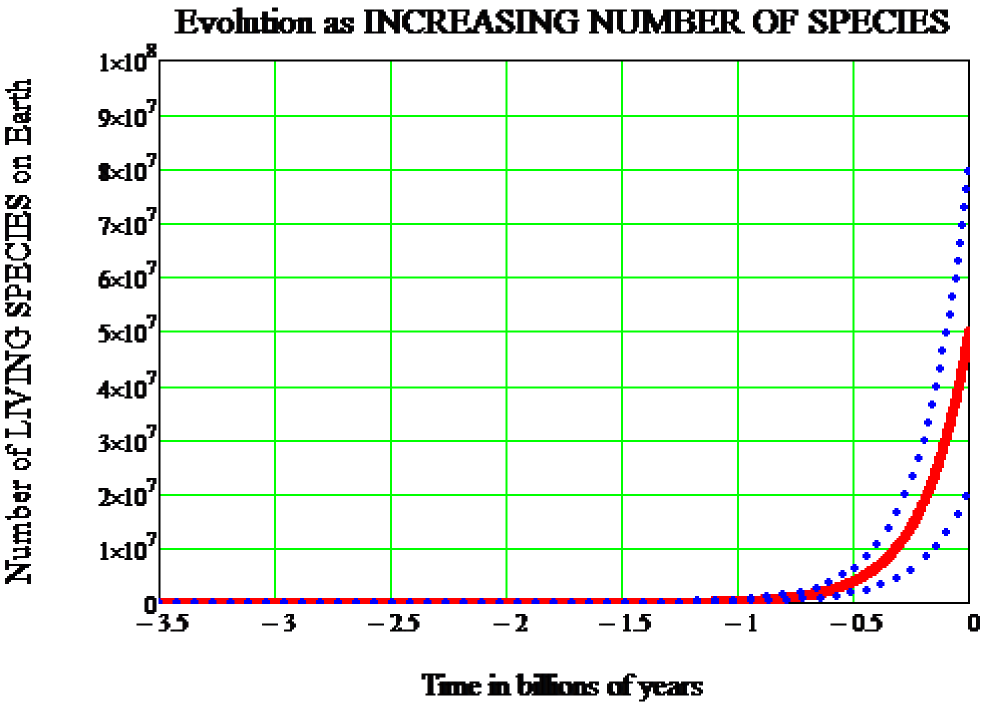

Fig. 1. BIOLOGICAL EVOLUTION as the increasing number of living Species on Earth between 3.5 billion years ago and now. The red solid curve is the mean value of the GBM stochastic process $L_{\text{GBM}}(t)$ given by (22), while the blue dot-dot curves above and below the mean value are the two standard deviation upper and lower curves, given by (11) and (12), respectively, with $m_{GBM}(t)$ given by (22). The "Cambrian Explosion" of life, that on Earth started around 542 million years ago, is evident in the above plot just before the value of -0.5 billion years in time, where all three curves start leaving the time axis and climbing up. Notice also that the starting value of living Species 3.5 billion years ago is ONE by definition, but it looks like zero in this plot since the vertical scale (which is the true scale here, not a log scale) does not show it. Notice finally that nowadays (i.e. at time $t = 0$) the two standard deviation curves have exactly the same distance from the middle mean value curve, i.e. 30 million living Species more or less the mean value of 50 million Species. These are assumed values that we used just to exemplify the GBM mathematics: biologists might assume other numeric values.

function, or stochastic process, with the notation $L(t)$ standing for "life at time t". In this paper we adopt the convention that capital letters represent random variables, i.e. stochastic processes if they depend on the time, while lower-case letters mean ordinary variables or functions.

1.2. Mean value of the lognormal process $l(t)$

The most important ordinary, continuous function of the time associated with a stochastic process like $L(t)$ is its mean value, denoted by

$$m_L(t) \equiv \langle L(t) \rangle \,. \tag{1}$$

The probability density function (pdf) of a stochastic process like $L(t)$ is assumed in Evo-SETI theory to be a b-lognormal, that is, its equation reads

$$L(t)_{\mathrm{pdf}}\,(n; M_L(t), \sigma, t) = \frac{e^{-\frac{[\ln(n)-M_L(t)]^2}{2\sigma_L^2(t-ts)}}}{\sqrt{2\pi}\quad \sigma_L\,\sqrt{t-ts}\quad n} \tag{2}$$

$$\text{with } \begin{cases} n & \geq 0 \\ t & \geq ts \end{cases} \text{ and } \begin{cases} \sigma_L & \geq 0 \\ M_L(t) & = \text{ arbitrary} \\ & \text{function of } t \end{cases}.$$

This assumption is in line with the extension in time of the statistical Drake equation, namely the foundational and statistical equation of SETI, as shown in (Maccone 2010). The real variable ts in (2) is the "time of start" of life on Earth. Finally about the notation: writing t_s would have been neater than ts but we could not do so since the Maxima symbolic manipulator, that we used to do all calculations, would interpret t_s as "the component s of the vector t" that is of course ***not*** what we want.

The mean value (1) is of course related to the pdf (2) by the relevant integral in the number n of living Species on Earth at time t, that is

$$m_L(t) \equiv \int_0^{\infty} n \frac{e^{-\frac{[\ln(n)-M_L(t)]^2}{2\sigma_L^2(t-ts)}}}{\sqrt{2\pi}\quad \sigma_L\,\sqrt{t-ts}\quad n} dn. \tag{3}$$

The "surprise" is that this integral (3) may be computed ***exactly*** with the key result that the mean value $m_L(t)$ is given by

$$m_L(t) = e^{M_L(t)} e^{\frac{\sigma_L^2}{2}(t-ts)}. \tag{4}$$

In turn, the last equation has the "surprising" property that it may be inverted ***exactly***, i.e. solved for $M_L(t)$:

$$M_L(t) = \ln\left(m_L(t)\right) - \frac{\sigma_L^2}{2}(t-ts). \tag{5}$$

1.3. $L(t)$ initial conditions at ts

Now about the initial conditions of the stochastic process $L(t)$, namely about the value $L(ts)$. We shall assume that the positive number

$$L(ts) = Ns > 0 \tag{6}$$

is always exactly known, i.e. with probability one:

$$Pr\{L(ts) = Ns\} = 1. \tag{7}$$

In the practice, Ns will be equal to 1 in the theories of evolution of life on Earth or on an exoplanet (i.e., there must have been a time ts in the past when the number of living Species was just one, let it be RNA or something else), and it will be equal to the number of living Species just before the asteroid/comet impact in the theories of mass extinction of life on a planet.

The mean value $m_L(t)$ of $L(t)$ also must equal the initial number Ns at the initial time ts, that is

$$m_L(ts) = Ns. \tag{8}$$

Replacing t by ts in (4), one then finds

$$m_L(ts) = e^{M_L(ts)} \tag{9}$$

that, checked against (8), immediately yields

$$m_L(ts) = e^{M_L(ts)} \text{ that is } M_L(ts) = \ln(Ns). \tag{10}$$

These are the initial conditions for the mean value.

After the initial instant ts, the stochastic process $L(t)$ unfolds oscillating above or below the mean value in an unpredictable way. Statistically speaking, however, we expect $L(t)$ "not to depart too much" from $m_L(t)$ and this fact is graphically shown in Fig. 1 by the two dot-dot blue curves above and below the mean value solid red curve $m_L(t)$. These two curves are the upper standard deviation curve

$$\text{upper\_std\_dev\_curve}\ (t) = m_L(t)\left[1 + \sqrt{e^{\sigma_L^2(t-ts)} - 1}\right] \tag{11}$$

and the lower standard deviation curve

$$\text{lower\_std\_dev\_curve}\ (t) = m_L(t)\left[1 - \sqrt{e^{\sigma_L^2(t-ts)} - 1}\right] \tag{12}$$

respectively (Proof: see Table 2 of Maccone 2014). Notice that both (11) and (12), at the initial time $t = ts$, equal the mean value $m(ts) = Ns$, that is, with probability one again, the initial value Ns is the same for all the three curves shown in Fig. 1. The function of the time

$$\text{coefficient\_of\_variation}\ (t) = \sqrt{e^{\sigma_L^2(t-ts)} - 1} \tag{13}$$

is called "coefficient of variation" by statisticians since the standard deviation of $L(t)$ (be careful: this is just the standard deviation $\Delta_L(t)$ of $L(t)$ and not either of the above two "upper" and "lower" standard deviation curves given by (11) and (12), respectively) is

$$\text{st\_dev\_curve}\ (t) \equiv \Delta_L(t) = m_L(t)\sqrt{e^{\sigma_L^2(t-ts)} - 1}. \tag{14}$$

Indeed, (14) shows that the coefficient of variation (13) is the ratio of $\Delta(t)$ to $m_L(t)$, i.e. it expresses how much the standard deviation "varies" with respect to the mean value. Having understood this fact, it is then obvious that the two curves (11) and (12) are obtained as

$$m_L(t) \pm \Delta_L(t) = m_L(t) \pm m_L(t)\sqrt{e^{\sigma_L^2(t-ts)} - 1} \tag{15}$$

respectively.

1.4. $L(t)$ final conditions at $te > ts$

Now about the final conditions for the mean value curve as well as for the two standard deviation curves. Let us call te the ending time of our mathematical analysis, namely the time beyond which we don't care any more about the values assumed by the stochastic process $L(t)$. In the practice, this te is zero (i.e. now) in the theories of evolution of life on Earth or on an exoplanet, or the time when the mass extinction ends (and life starts growing up again) in the theories of mass extinction of life on a planet. First of all, it is clear that, in full analogy to the initial condition (8) for the mean value, also the final condition has the form

$$m_L(te) = Ne > 0 \tag{16}$$

where Ne is a positive number denoting the number of Species alive at the end time te. But we don't know what random value will $L(te)$ take. We only know that its standard deviation curve (14) will take at that time te a certain positive value that will differ by a certain amount from the mean value (16). In other words, we only know from (14) that one has

$$\delta Ne = \Delta_L(te) = m_L(te)\sqrt{e^{\sigma_L^2(te-ts)} - 1}. \tag{17}$$

Dividing (17) by (16) the common factor $m_L(te)$ disappears, and one is left with

$$\frac{\delta Ne}{Ne} = \sqrt{e^{\sigma_L^2(te-ts)} - 1}. \tag{18}$$

Solving this for σ_L finally yields

$$\sigma_L = \frac{\sqrt{\ln\left[1 + \left(\frac{\delta Ne}{Ne}\right)^2\right]}}{\sqrt{te - ts}}. \tag{19}$$

This equation expresses the so far unknown numerical parameter σ_L in terms of the initial time ts plus the three final-time parameters $(te, Ne, \delta Ne)$. Thus, in conclusion, we have shown that, once the five parameters $(ts, Ns, te, Ne, \delta Ne)$ are assigned numerically, the lognormal stochastic process $L(t)$ is determined completely. Finally notice that the square of (19) may be rewritten in the following different form:

$$\sigma_L^2 = \frac{\ln\left[1 + \left(\frac{\delta Ne}{Ne}\right)^2\right]}{te - ts} = \ln\left\{\left[1 + \left(\frac{\delta Ne}{Ne}\right)^2\right]^{\frac{1}{te-ts}}\right\} \tag{20}$$

from which we infer the formula

$$e^{\sigma_L^2} = e^{\ln\left\{\left[1+\left(\frac{\delta Ne}{Ne}\right)^2\right]^{\frac{1}{te-ts}}\right\}} = \left[1 + \left(\frac{\delta Ne}{Ne}\right)^2\right]^{\frac{1}{te-ts}}. \tag{21}$$

This equation (21) enables us to get rid of $e^{\sigma_L^2}$ replacing it by virtue of the four boundary parameters supposed to be known: $(ts, te, Ne, \delta Ne)$. It will be later used in Section 1.8 in order to rewrite the Peak-Locus Theorem in terms of the boundary conditions, rather than in terms of $e^{\sigma_L^2}$.

1.5. Important special cases of $m_L(t)$

1. The particular case of (1) when the mean value $m_L(t)$ is given by the generic exponential

$$m_{\text{GBM}}(t) = N_0 e^{\mu_{\text{GBM}} t} = \text{or, alternatively,} = A e^{Bt} \tag{22}$$

is called Geometric Brownian Motion (GBM), and is widely used in financial mathematics, where it represents the "underlying process" of the stock values (Black-Sholes models). This author used the GBM in his previous models of Evolution and SETI (refs. Maccone 2010, Maccone 2011, Maccone (2012), Maccone 2014), since it was assumed that the growth of the number of ET civilizations in the Galaxy, or, alternatively, the number of living Species on Earth over the last 3.5 billion years, ***grew exponentially*** (Malthusian growth). Notice that, upon equating the two right-hand-sides of (4) and (21), we find

$$e^{M_{\text{GBM}}(t)} e^{\frac{\sigma_{\text{GBM}}^2}{2}(t-ts)} = N_0 e^{\mu_{\text{GBM}}(t-ts)}. \tag{23}$$

Solving this equation for $M_{\text{GBM}}(t)$ yields

$$M_{\text{GBM}}(t) = \ln N_0 + \left(\mu_{\text{GBM}} - \frac{\sigma^2_{\text{GBM}}}{2}\right)(t - ts) \tag{24}$$

This is (with $ts = 0$) just the "mean value showing at the exponent" of the well-known GBM pdf, i.e.

$$\text{GBM}(t)\_pdf\,(n; N_0, \mu, \sigma, t) = \frac{e^{-\frac{\left[\ln(n)-\left(\ln N_0+\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right]^2}{2\sigma^2 t}}}{\sqrt{2\pi}\sigma\,\sqrt{tn}}. \tag{25}$$

We conclude this short description of the GBM as the exponential sub-case of the general lognormal process (2) by warning that "GBM" is a misleading name, since GBM is a lognormal process and not a Gaussian one, as the Brownian Motion is indeed in physics.

2. As we mentioned already, another interesting particular case of the mean value function $m_L(t)$ in (1) is when it equals a generic ***polynomial in*** t ***starting at*** ts, namely

$$m_{\text{polynomial}}(t) = \sum_{k=0}^{\text{polynomial degree}} c_k(t - ts)^k \tag{26}$$

with c_k being the coefficient of the k-th power of the time t in the polynomial (26). We just confine ourselves to mention that the case where (26) is a second-degree polynomial (i.e. a parabola in t) may be used to describe the Mass Extinctions on Earth over the last 3.5 billion years (see Maccone 2017a).

3. We must also introduce the notion of b-lognormal

$$\text{b-lognormal\_pdf}(t; \mu, \sigma, b) = \frac{e^{-\frac{[\ln(t-b)-\mu]^2}{2\sigma^2}}}{\sqrt{2\pi}\quad\sigma\quad(t-b)} \tag{27}$$

holding for $t \geq b$, and meaning the lifetime of a living being, let it be a cell, a plant, a human, a civilization of humans, or even an ET civilization (Maccone (2014), in particular pages 227-245).

1.6. Boundary conditions when $m(t)$ is a first, second or third degree polynomial in the time $(t - ts)$

In Maccone (2017a) the reader may find a mathematical model of Biological Evolution different from the GBM model described in terms of GBMs. That is the Markov-Korotayev model, for which this author proved the mean value (1) to be a Cubic(t) i.e. a third degree polynomial in t.

We summarize hereafter the key formulae proven in Maccone (2017a) about the case when the assigned mean value $m_L(t)$ is a polynomial in t starting at ts, that is:

$$m_L(t) = \sum_{k=0}^{3} c_k(t - ts)^k. \tag{28}$$

1. ***the mean value is a straight line.*** Then this straight line simply is the line through the two points (ts, Ns) and (te, Ne), that, after a few rearrangements, turns out to be:

$$m_{\text{straight\_line}}(t) = (Ne - Ns)\frac{t - ts}{te - ts} + Ns. \tag{29}$$

2. ***The mean value is a parabola***, i.e. a quadratic polynomial in t. Then, the equation of such a parabola reads :
$$m_{\text{parabola}}(t) = (Ne - Ns)\frac{t - ts}{te - ts}\left[2 - \frac{t - ts}{te - ts}\right] + Ns. \tag{30}$$
Equation (30) was actually firstly derived by this author in Maccone (2017a), pages 299-301, in relationship to Mass Extinctions (i.e. it is a decreasing function of the time).
3. ***The mean value is a cubic***. Then, in Maccone (2017a), pages 304-307 this author proved, in relation to the Markov-Korotayev model of Evolution, that the ***cubic*** mean value of the $L(t)$ lognormal stochastic process is given by the cubic equation in t
$$m_{\text{cubic}}(t) = (Ne - Ns)\frac{(t - ts)\left[2(t - ts)^2 - 3(t_{\text{Max}} + t_{\text{min}} - 2ts)(t - ts) + 6(t_{\text{Max}} - ts)(t_{\text{min}} - ts)\right]}{(te - ts)\left[2(te - ts)^2 - 3(t_{\text{Max}} + t_{\text{min}} - 2ts)(t - ts) + 6(t_{\text{Max}} - ts)(t_{\text{min}} - ts)\right]} + Ns \tag{31}$$
Notice that, in (31) one has, in addition to the usual initial and final conditions $Ns = m_L(ts)$ and $Ne = m_L(te)$, two more "middle conditions" referring to the two instants (t_M, t_m) at which the Maximum and the minimum of the $cubic(t)$ occur, respectively:
$$\begin{cases} t_{\text{min}} & = \text{time\_of\_the\_Cubic\_minimum} \\ t_{\text{max}} & = \text{time\_of\_the\_Cubic\_maximum} \end{cases}. \tag{32}$$

1.7. Peak-locus theorem

The Peak-Locus theorem is a new mathematical discovery of ours playing a central role in Evo-SETI. In its most general formulation, it holds good for any lognormal process $L(t)$ and any arbitrary mean value $m_L(t)$. In the GBM case, it is shown in Figure 2.

The Peak-Locus theorem states that the family of b-lognormals each having its peak exactly located ***upon*** the mean value curve (1), is given by the following three equations, specifying the parameters $\mu(p)$, $\sigma(p)$ and $b(p)$, appearing in (27) as three functions of the independent variable p, the b-lognormal's peak: that is, if rewritten directly in terms of $m_L(p)$:

$$\begin{cases} \mu(p) & = \frac{e^{\sigma_L^2 p}}{4\pi\left[m_L(p)\right]^2} - p\,\frac{\sigma_L^2}{2} \\ \sigma(p) & = \frac{e^{\frac{\sigma_L^2}{2} p}}{\sqrt{2\pi}\, m_L(p)} \\ b(p) & = p - e^{\mu(p) - \left[\sigma(p)\right]^2} \end{cases}. \tag{33}$$

The proof of (33) is lengthy and was given as a special pdf file (written in the language of the Maxima symbolic manipulator) that the reader may freely download in the web site of Maccone (2017a).

But we now present an important new result: the Peak-Locus Theorem (33) rewritten not in terms of σ_L anymore, but rather in terms of the four boundary parameters supposed to be known: $(ts, te, Ne, \delta Ne)$. To this end, we must insert (21) and (20) into (33), with the result

$$\begin{cases} \mu(p) & = \frac{\left[1+\left(\frac{\delta Ne}{Ne}\right)^2\right]^{\frac{p}{te-ts}}}{4\pi\left[m_L(p)\right]^2} - \ln\left\{\left[1 + \left(\frac{\delta Ne}{Ne}\right)^2\right]^{\frac{p}{2(t-ts)}}\right\} \\ \sigma(p) & = \frac{\left[1+\left(\frac{\delta Ne}{Ne}\right)^2\right]^{\frac{p}{2(t-ts)}}}{\sqrt{2\pi}\, m_L(p)} \\ b(p) & = p - e^{\mu(p) - \left[\sigma(p)\right]^2}. \end{cases} \tag{34}$$

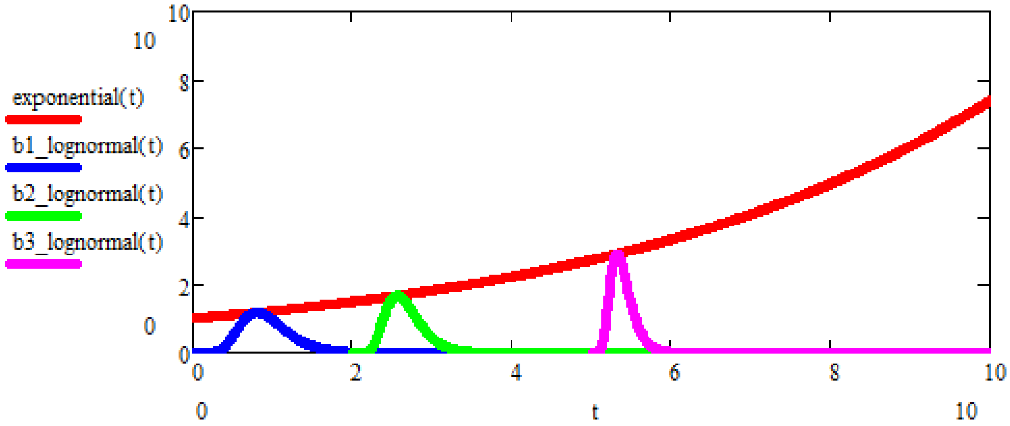

Fig. 2. Biological Exponential as the geometric LOCUS OF THE PEAKS of b-lognormals for the GBM case. Each b-lognormal is a lognormal starting at a time ($t = b =$ birth time) and represents a different Species that originated at time b of the Biological Evolution. This is CLADISTICS, as seen through the glasses of our Evo-SETI model. It is evident that, when the generic "Running b-lognormal" moves to the right, its peak becomes higher and higher and narrower and narrower, since the area under the b-lognormal always equals 1. Then, the (Shannon) ENTROPY of the running b-lognormal is the DEGREE OF EVOLUTION reached by the corresponding Species (or living being, or a civilization, or an ET civilization) in the course of Evolution (see, for instances, Felsenstein 2004; Nei 2000, 2013; Wiki 2018).

In the particular GBM case, the mean value is (22) with $\mu_{\text{GBM}} = B$, $\sigma_L = \sqrt{2B}$ and $N_0 = Ns = A$. Then, the Peak-Locus theorem (33) with $ts = 0$ yields:

$$\begin{cases} \mu(p) & = \frac{1}{4\pi A^2} - B\,p, \\ \sigma & = \frac{1}{\sqrt{2\pi}A}, \\ \ b(p) & = p - e^{\mu(p)-\sigma^2} \end{cases} . \tag{35}$$

In this simpler form, the Peak-Locus theorem was already published by the author in Maccone (2011), Maccone (2012) and Maccone (2014), while its most general forms are (33) and (34).

1.8. EvoEntropy(p) as measure of evolution

The (Shannon) Entropy of the b-lognormal (27) is

$$H(p) = \frac{1}{\ln(2)}\left[\ln\left(\sqrt{2\pi}\,\sigma(p)\right) + \mu(p) + \frac{1}{2}\right]. \tag{36}$$

This is a function of the peak abscissa p and is measured in bits, as in Shannon's Information Theory. By virtue of the Peak-Locus Theorem (33), it becomes

$$H(p) = \frac{1}{\ln(2)}\left\{\frac{e^{\sigma_L^2 p}}{4\pi[m_L(p)]^2} - \ln(m_L(p)) + \frac{1}{2}\right\} \tag{37}$$

One may also rewrite (37) directly in terms of the four boundary parameters $(ts, te, Ne, \delta Ne)$ upon inserting (21) into (37), with the result

$$H(p) = \frac{1}{\ln(2)} \left\{ \frac{\left[1 + \left(\frac{\delta Ne}{Ne}\right)^2\right]^{\frac{p}{te-ts}}}{4\pi [m_L(p)]^2} - \ln(m_L(p)) + \frac{1}{2} \right\} \tag{38}$$

Thus, (37) or (38) yield the Entropy of each member of the family of ∞^1 b-lognormals (the family's parameter is p) peaked ***upon*** the mean value curve (1). The b-lognormal Entropy (37) or (38) is thus the Measure of the Amount of Evolution of that b-lognormal: it measures the decreasing disorganization in time of what that b-lognormal represents, let it be a cell, a plant, a human or even a civilization. Entropy is thus disorganization decreasing in time. However, one would prefer to use a measure of the increasing organization in time. The EvoEntropy of p

$$\text{EvoEntropy}(p) = -[H(p) - H(ts)] \tag{39}$$

(Entropy of Evolution) is a function that has a minus sign in front, thus changing the decreasing trend of the (Shannon) Entropy (38) into the increasing trend of our EvoEntropy (39). This need to change the sign in front of (38) is actually an old story started in 1944 by the famous physicist Erwin Schrödinger and continued in 1953 by the French-American physicist León Brillouin when he coined the word ***Negentropy*** to mean entropy with the reversed sign. More at the site `https://en.wikipedia.org/wiki/Negentropy`

The EvoEntropy (39) that is

$$\text{EvoEntropy}(p) = -[H(p) - H(ts)] \tag{40}$$

(Entropy of Evolution) is a function of p that has a minus sign in front, thus changing the decreasing trend of the (Shannon) Entropy (36) into the increasing trend of our EvoEntropy (40). In addition, our EvoEntropy starts at zero at the initial time ts, as expected:

$$\text{EvoEntropy}(ts) = 0. \tag{41}$$

By virtue of (37), and keeping (8) in mind, the EvoEntropy (40) becomes

$$\begin{aligned} &\text{EvoEntropy}(p)\_\text{of}\_\text{the}\_\text{Lognormal}\_\text{Process}\_L(t) = \\ &= \frac{1}{\ln(2)} \left\{ \frac{e^{\sigma_L^2\, ts}}{4\pi N s^2} - \frac{e^{\sigma_L^2\, p}}{4\pi [m_L(p)]^2} + \ln\left(\frac{m_L(p)}{Ns}\right) \right\}. \end{aligned} \tag{42}$$

Alternatively, we may rewrite (42) directly in terms of the five boundary parameters $(ts, Ns, te, Ne, \delta Ne)$ upon inserting (21) into (42), thus finding

$$\begin{aligned} &\text{EvoEntropy}(p)\_\text{of}\_\text{the}\_\text{Lognormal}\_\text{Process}\_L(t) = \\ &= \frac{1}{\ln(2)} \left\{ \frac{\left[1 + \left(\frac{\delta Ne}{Ne}\right)^2\right]^{\frac{ts}{te-ts}}}{4\pi N s^2} - \frac{\left[1 + \left(\frac{\delta Ne}{Ne}\right)^2\right]^{\frac{p}{te-ts}}}{4\pi [m_L(p)]^2} + \ln\left(\frac{m_L(p)}{Ns}\right) \right\}. \end{aligned} \tag{43}$$

Let us now remark that the standard deviation at the end time, δNe, really is irrelevant to compute the EvoEntropy (43). In fact, the EvoEntropy (40) is just a continuous ***curve***, and not a stochastic process. So, we may assign δNe the numeric value of zero, and get the EvoEntropy curve. Keeping this in mind, we see that the true EvoEntropy curve (43) is obtained by squashing down (43) into the mean value curve $m_L(t)$, and that only happens if we let

$$\delta Ne = 0 \tag{44}$$

Inserting (44) into (43), the latter simplifies dramatically into

$$\begin{aligned}&\text{EvoEntropy}(p)\_\text{of\_the\_Lognormal\_Process\_}L(t) = \\ &= \frac{1}{\ln(2)}\left\{\frac{1}{4\pi Ns^2} - \frac{1}{4\pi[m_L(p)]^2} + \ln\left(\frac{m_L(p)}{Ns}\right)\right\}\end{aligned} \tag{45}$$

which is the final form of the EvoEntropy (42) and (43) that we will use in the sequel. We may now see very neatly that the final EvoEntropy (45) is made up by three terms:

1. The constant term
$$\frac{1}{4\pi Ns^2} \tag{46}$$
whose numeric value in the particularly important case $Ns = 1$ boils down to
$$\frac{1}{4\pi} = 0.07958 \tag{47}$$
that is almost zero.
2. The inverse square term
$$-\frac{1}{4\pi\,[m_L(p)]^2} \tag{48}$$
that rapidly falls down to zero for $m_L(t)$ approaching infinity. In other words, this inverse-square term may become "almost negligible" for large values of the time p.
3. And finally the dominant term, i.e. the logarithmic term
$$\ln\left(\frac{m_L(p)}{Ns}\right) \tag{49}$$
that actually is the leading term in the EvoEntropy (45) for large values of the time p. In conclusion, the EvoEntropy (45) in essence depends basically upon its logarithmic term (49) and so its shape in time must be similar to the shape of a logarithm i.e. nearly vertical at the beginning of the curve and then progressively approaching the horizontal shape. This curve has no maxima nor minima, nor inflexions.

1.9. Perfectly linear evoentropy when the mean value is perfectly exponential (gbm): this is just the molecular clock!

In the GBM case (22), that is when the mean value is given by the exponential

$$m_{GBM}(t) = Ns\, e^{\frac{\sigma_L^2}{2}(t-ts)} = Ns\, e^{B(t-ts)} \tag{50}$$

the EvoEntropy (42) becomes just an ***exact linear function of the time*** p since the first two terms inside the braces in (42) just ***cancel*** against each other.

Proof: just insert (50) into (42) and then simplify:

$$\begin{aligned}
&\text{EvoEntropy}(p)_\text{of_GBM} = \\
&= \frac{1}{\ln(2)} \left\{ \frac{e^{\sigma_L^2\, ts}}{4\pi N s^2} - \frac{e^{\sigma_L^2\, p}}{4\pi \left[N s\, e^{\frac{\sigma_L^2}{2}(p-ts)} \right]^2} + \ln\left(\frac{N s\, e^{\frac{\sigma_L^2}{2}(p-ts)}}{N s} \right) \right\} = \\
&= \frac{1}{\ln(2)} \left\{ \frac{e^{\sigma_L^2\, ts}}{4\pi N s^2} - \frac{e^{\sigma_L^2\, p}}{4\pi N s^2\, e^{\sigma_L^2 (p-ts)}} + \ln\left(e^{\frac{\sigma_L^2}{2}(p-ts)} \right) \right\} = \\
&= \frac{1}{\ln(2)} \left\{ \frac{e^{\sigma_L^2\, ts}}{4\pi N s^2} - \frac{1}{4\pi N s^2\, e^{\sigma_L^2 (-ts)}} + \frac{\sigma_L^2}{2}(p-ts) \right\} = \\
&= \frac{1}{\ln(2)} \left\{ \frac{\sigma_L^2}{2}(p-ts) \right\} = \frac{B}{\ln(2)} \cdot (p-ts)\,.
\end{aligned} \tag{51}$$

In other words, the GBM EvoEntropy is given by

$$\text{GBM\_EvoEntropy}\,(p) = \frac{B}{\ln(2)} \cdot (p - ts)\,. \tag{52}$$

This is, of course, a straight line in the time p starting at the time ts of the Origin of Life on Earth and ***increasing linearly thereafter. It is measured in bits/individual and is shown in Figure 3***.

But THIS IS THE SAME LINEAR BEHAVIOUR IN TIME AS THE MOLECULAR CLOCK, that is the technique in molecular evolution that uses fossil constraints and rates of molecular change to deduce the time in geologic history when two Species or other taxa diverged. The molecular data used for such calculations are usually nucleotide sequences for DNA or amino acid sequences for proteins (see Maccone 2015, 2017b). So, we have discovered that the Entropy in our Evo-SETI model and the Molecular Clock are the same linear time function, apart for multiplicative constants (depending on the adopted units, like bits, seconds, etc.). This conclusion appears to be of key importance to understand ”where a newly discovered exoplanet stands on its way to develop LIFE”.

1.10. Introducing ”ee” (= earth evolution) as our evo-seti unit: information equal to the evo-entropy reached by the evolution of life on earth nowadays

The purpose of this paper really is to propose our new Evo-SETI UNIT of evolution.

From the above discussion it follows that the numeric value of the Evo-SETI unit is about 25.575 bits if life on Earth started 3.5 billion years ago.

This unit we propose to call EE (EarthEvolution), so that we define EE as

$$\begin{aligned}
&EE = \\
&\quad = \text{Evo_SETI_unit_of_Evolution_as_the_} \\
&\quad \text{difference_in_Information_Content_} \\
&\quad \text{between_RNA_and_Humans} = \\
&\quad = 25.575 \text{ bits.}
\end{aligned} \tag{53}$$

Thus, a planet like Mars will have an Evo-SETI evolution much less than $1EE$ in case it hosted any primitive form of life, even in the past. And just $0EE$ in case it did not host any form of life at all.

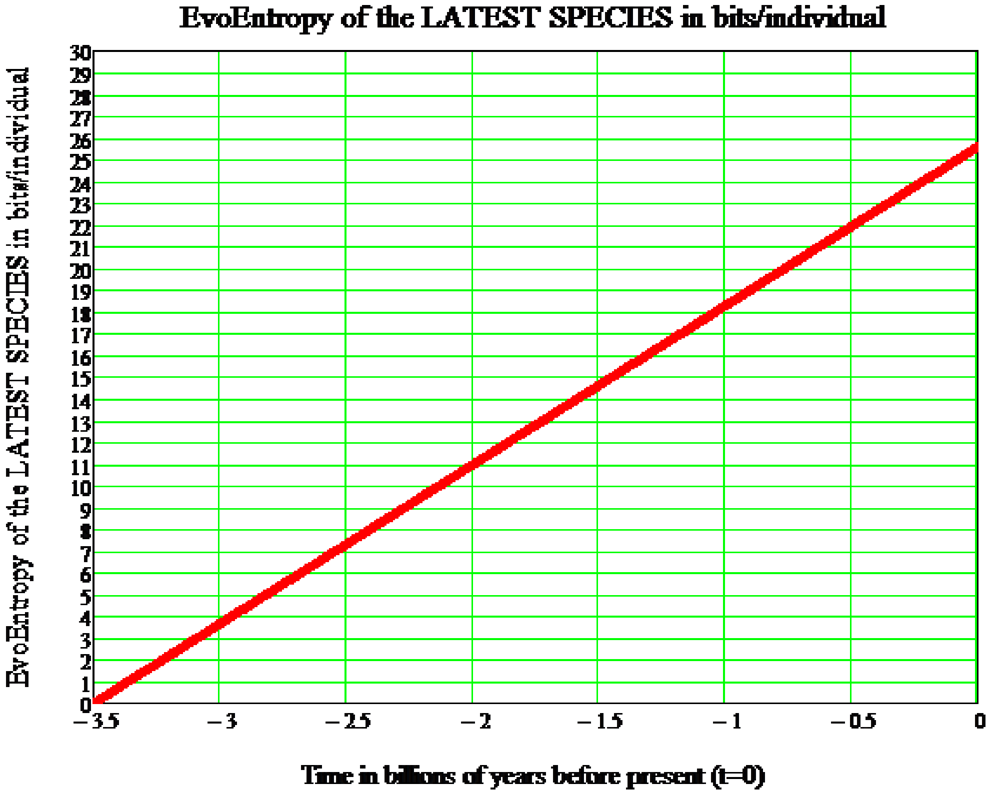

Fig. 3. EvoEntropy (in bits per individual) of the latest Species appeared on Earth during the last 3.5 billion years if the mean value is an increasing exponential, i.e. if our lognormal stochastic process $L(t)$ is a GBM. This straight line shows that a Man (nowadays) is 25.575 bits more evolved than the first form of life (RNA) that started 3.5 billion years ago.

On the contrary, an exoplanet hosting an ExtraTerrestrial Civilization much more advanced than the Human one will have a EE larger or much larger than 1 according to the higher degree of Evolution reached by that Civilization when compared to Humans.

This is our Evo-SETI SCALE quantifying the Evolution of Life in the Universe and, at the same time, proving once again that the Molecular Clock discovered in 1962 by Emil Zuckerkandl (1922-2013) and Linus Pauling (1901-1994) is indeed a correct, fundamental law of nature.

1.11. Conclusions about the molecular clock as a part of evo-seti theory

The Biological Evolution of life on Earth over the last 3.5 to roughly 4 billion years has hardly been cast into any ”profound” mathematical form. The molecular clock is an exception in that Zuckerkandl and Pauling cast it in a ”straight line” form, i.e. in the easiest possible geometrical form. Since 2012 this author has tried to do profound mathematics about the evolution of life on earth by resorting to lognormal probability distributions in the time, starting each at a different time instant b (birth) and called b-lognormals (refs. Maccone 2010, Maccone 2011, Maccone 2012, Maccone 2014, Maccone 2017a and Maccone 2017b). His discovery (in the years 2010-

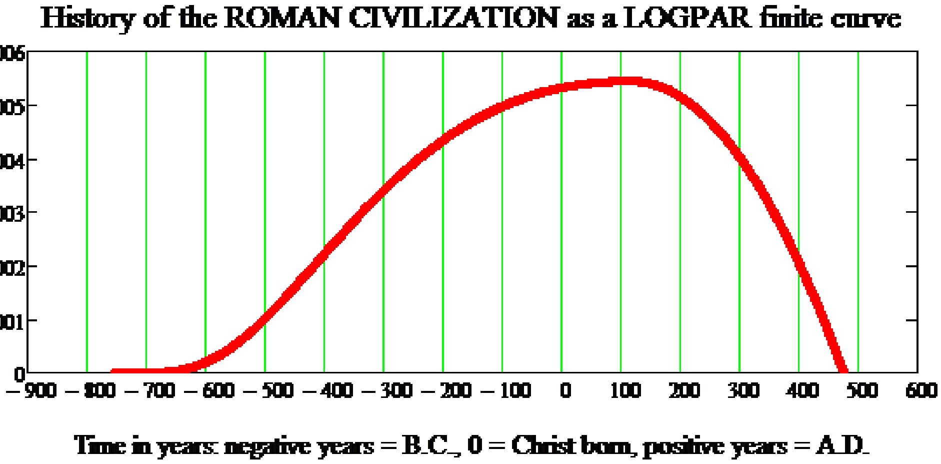

Fig. 4. Representation of the History of the Roman civilization as a LOGPAR finite curve. Rome was funded in 753 B.C., i.e. in the year -753 in our notation, or $b = -753$. Then the Roman republic and empire (the latter since the first emperor, Augustus, roughly after 27 B.C.) kept growing in conquered territory until it reached its peak (maximum extension, up to Susa in current Iran) in the year 117 A.D., i.e. $p = 117$, under emperor Trajan. Afterwards it started to decline and loose territory until the final collapse in 476 A.D. ($d = 476$, Romulus Augustulus, last emperor). Thus, just three points in time are necessary and sufficient to summarize the History of Rome: $b = -753$, $p = 117$, $d = 476$. No other intermediate point, like senility in between peak and death, is necessary at all since we now used a logpar rather than a b-lognormal, as this author had done prior to 2017. The numbers along the vertical axis will be explained later.

2015) of the Peak-Locus Theorem valid for any enveloping mean value (and not just the exponential one, GBM) has made it possible the use of the Shannon Entropy of Information Theory as the correct mathematical tool measuring the Evolution of life in bits/individual. In conclusion, what happened on Earth over the last 4 billion years is now summarized by a few simple statistical equations, but is just about the evolution of life on Earth, and not on other Exoplanets. The extension of our Evo-SETI Theory to life on other Exoplanets will be possible only when SETI, the current scientific Search for ExtraTerrestrial Intelligence, will achieve the first Contact between Humans and an Alien Civilization.

2. Lifetime = a logpar power curve with known energy

2.1. Introduction to logpar finite lifetime curves

The following new idea is easy: we seek to represent any lifetime by virtue of just three points in time: birth, peak, death (BPD). No other point in between. That is, no other ”senility point” s as those appearing in all b-lognormals like (27) that this author had published in his Evo-SETI Theory prior to 2017. In fact, it is easier and more natural to describe someones lifetime just in terms of birth, peak and death, than in terms of birth, senility and death, because when the senility arrives is rather ***uncertain*** and so hard to define in the practice for any individual.

Please look at Figure 4. The first part, the one on the left, i.e. prior to the peak time p, is just a b-lognormal: it starts at birth time b, climbs up to the adolescence time a (ascending inflexion point of the b-lognormal) (in reality the adolescence time should more properly be called ”puberty time” since it marks the beginning of the reproduction capacity for that individual) and finally reaches the peak time at p (maximum, i.e. the point of zero first derivative of the b-lognormal). All this is just ordinary b-lognormal stuff, as we have been ”preaching” since about 2012.

But now the novelty comes, i.e. the second part, the one on the right: that is just a ***parabola*** having its vertex exactly at the peak time p. Notice that this definition automatically implies that the tangent line at the peak is horizontal, i.e. the same for both the b-lognormal and the parabola. Notice also that, after the peak, the parabola plunges down until it reaches the time axis at the death time d. Therefore this new definition of death time d is different from the old definition of d applying to b-lognormals alone, as we did prior to 2017. ***And this is the LOGPAR (b-LOGnormal plus PARabola) new CURVE FINITE IN TIME (namely ranging in time just between birth and death). We introduce the logpar for the first time in the present paper and we study it with surprising results***.

2.2. Finding the parabola equation of the right part of the logpar

We shall now cast into appropriate mathematics the above popular description of what a logpar curve is.

Consider the equation of a parabola in the time t having vertical axis along the $t = p$ vertical line:

$$y = \alpha\,(t-p)^2 + \beta\ (t-p) + \gamma \tag{54}$$

where α, β and γ are the three coefficients of the time that we must determine according to the assumptions shown in Figure 4. To find them, we must resort to the three conditions that we know to hold by virtue of a glance to Figure 4:

1. CONDITION: the height of the peak is P, just the same as the height of the peak of the b-lognormal on the left in Figure 4. Thus, inserting the two equations of the peak, namely

   $$\begin{cases} t = p \\ y = P \end{cases} \tag{55}$$

   into (54), the latter yields immediately

   $$P = \gamma \tag{56}$$

   that, when inserted back into (54), changes it into

   $$y = \alpha\,(t-p)^2 + \beta\ (t-p) + P. \tag{57}$$

2. CONDITION: the tangent straight line at both the b-lognormal and the parabola at the peak abscissa p is horizontal. In other words, the first derivative of (57) at $t = p$ must equal zero. Differentiating (57) with respect to t, equalling that to zero and then solving for β yields

   $$\beta = -2\,\alpha\ (t-p)\,. \tag{58}$$

   Inserting (58) into (57), the latter is turned into

   $$y = -\,\alpha\,(t-p)^2 + P. \tag{59}$$

3. CONDITION: at the death time d, one must have $y = 0$, yielding from (59) the equation

   $$0 = -\,\alpha\,(d-p)^2 + P. \tag{60}$$

Solving (60) for α one gets

$$\alpha = \frac{P}{(d-p)^2}. \tag{61}$$

Finally, inserting (61) into (59) the desired equation of the parabola is found

$$y(t) = P\left[1 - \frac{(t-p)^2}{(d-p)^2}\right]. \tag{62}$$

As confirmation, one may check that (62) immediately yields the two conditions

$$\begin{cases} y(p) = P \\ y(d) = 0 \end{cases}. \tag{63}$$

2.3. Area under theb-lognormal curve on the left part of the logpar between birth and peak

As for the b-lognormal between birth and peak, making up the left part of the logpar curve, we already know all its mathematical details from the previous many papers published by this author on this topics, but we shall summarize here the main equations for the sake of completeness.

The equation of the b-lognormal starting at b is (27) and reads

$$\text{b\_lognormal}(t;\mu,\sigma,b) = \frac{e^{-\frac{(\log(t-b)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma\ (t-b)}. \tag{64}$$

Tables listing the main equations that can be derived from (64) were given by this author in Maccone (2012) and Maccone (2014) and we shall not re-derive them here again. We just confine ourselves to reminding that:

1. The abscissa p of the peak of (64) is given by

$$p = b + e^{\mu-\sigma^2}. \tag{65}$$

Proof. Take the derivative of (64) with respect to t and set it equal to zero. Then solve the resulting equation for t, that now becomes p, and (65) is found.

2. The ordinate P of the peak of (64) is given by

$$P = \frac{e^{\frac{\sigma^2}{2}-\mu}}{\sqrt{2\pi}\,\sigma}. \tag{66}$$

Proof. Rewrite p instead of t in (64) and then insert (65) instead of p. Then simplify to get (66).

3. The abscissa of the adolescence point (that should actually be better named "puberty point") is the abscissa of the ascending inflexion point of (64). It is given by

$$a = b + e^{-\frac{\sigma\sqrt{\sigma^2+4}}{2}-\frac{3\sigma^2}{2}+\mu}. \tag{67}$$

Proof. Take the second derivative of (64) with respect to t and set it equal to zero. Then solve the resulting equation for t that now becomes a and (67) is found.

4. The ordinate of the adolescence point is given by

$$\frac{e^{-\frac{\sigma\sqrt{\sigma^2+4}}{4}+\frac{\sigma^2}{4}-\mu-\frac{1}{2}}}{\sqrt{2\pi}\sigma}. \tag{68}$$

Proof. Just rewrite a instead of t in (64) and then insert (67) and simplify the result.
Let us now notice that, within the framework of the logpar theory described in this paper, we may NOT say that (64) fulfills the normalization condition

$$\int_b^\infty \text{b\_lognormal}(t; \mu, \sigma, b)\, dt = 1 \tag{69}$$

since t here is only allowed to range between b and p. Rather than adopting (69), we must thus replace (69) by the integral of (64) between b and p only. Fortunately, it is possible to evaluate this integral in terms of the ***error function*** defined by

$$erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{70}$$

In fact, the integral of the b-lognormal (64) between b and p turns out to be given by

$$\int_b^p \text{b\_lognormal}(t; \mu, \sigma, b)\, dt =$$

$$= \int_b^p \frac{e^{-\frac{(\log(t-b)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma\ (t-b)} dt =$$

$$= \frac{1 + erf\left(\frac{\sqrt{2}\ln(p-b)-\sqrt{2}\mu}{2\,\sigma}\right)}{2} =$$

Now, inserting (65) instead of p into the last erf argument, a remarkable simplification occurs: μ and p both disappear and only σ is left. In addition, the erf property $erf(-x) = -erf(x)$ allows us to rewrite

$$= \frac{1 + erf\left(-\frac{\sigma}{\sqrt{2}}\right)}{2} = \frac{1 - erf\left(\frac{\sigma}{\sqrt{2}}\right)}{2}. \tag{71}$$

In conclusion, the area under the b-lognormal between birth and peak is given by

$$\int_b^p \text{b\_lognormal}(t; \mu, \sigma, b)\, dt = \frac{1 - erf\left(\frac{\sigma}{\sqrt{2}}\right)}{2}. \tag{72}$$

This result will prove to be of key importance for the further developments described in the present paper.

2.4. Area under theb-lognormal curve on the right part of the logpar between birth and peak and death

We already proved that the parabola on the right part of the logpar curve has the equation (62). Now we want to find the area under this parabola between peak and death, that is

$$\begin{aligned}
&\int_p^d \left\{ P\left[1 - \frac{(t-p)^2}{(d-p)^2}\right]\right\} dt = \\
&= P\,(d-p) - \frac{P}{(d-p)^2}\int_p^d (t-p)^2\, dt = \\
&= P\,(d-p) - \frac{P}{(d-p)^2}\,\frac{(d-p)^3}{3} = \\
&= \frac{2\,P\,(d-p)}{3} = \\
&= \frac{1}{2}\cdot\left[\frac{4}{3}P\,(d-p)\right] = \frac{1}{2}\cdot\begin{bmatrix} Area\ of\ Archimedes \\ parabolic\ segment, \\ proved\ < 212\ B.C. \end{bmatrix}.
\end{aligned} \tag{73}$$

The great Ancient Greek mathematician Archimedes (circa 287 B.C. - 212 B.C.) of Syracuse (Sicily) already "knew" the last integral result more than 1870 years before the Calculus was discovered by Newton and Leibniz after 1660. Actually, the first mathematician to rediscover (73) in modern times was Bonaventura Cavalieri (1598-1647) in 1635 (see `https://en.wikipedia.org/wiki/Cavalieri%27s_quadrature_formula`). Archimedes had used Eudoxus' "method of exhaustion" to compute the area of a segment of the parabola, as neatly described at the site `https://en.wikipedia.org/wiki/The_Quadrature_of_the_Parabola`. In conclusion, the area under our parabola between peak and death is given by (73), that we now rewrite as

$$\int_p^d \left\{ P\left[1 - \frac{(t-p)^2}{(d-p)^2}\right]\right\} dt = \frac{2\,P\,(d-p)}{3}. \tag{74}$$

2.5. Area under the full logpar curve between birth and death

We are now in a position to compute the full area A under the logpar curve, that is given by the ***sum*** of equations (72) and (74), that is

$$\frac{1 - erf\left(\frac{\sigma}{\sqrt{2}}\right)}{2} + \frac{2\,P\,(d-p)}{3} = A. \tag{75}$$

This is the most important equation in this paper. In fact, if we want the logpar be a truly probability density function (pdf), we must assume in (75)

$$A = 1. \tag{76}$$

But, surprisingly, we shall NOT do so! Let us rather ponder over what we are doing:

1. We are creating a "Mathematical History" model where the "unfolding History" of each Civilization in the time is represented by a logpar curve.

2. The knowledge of only three points in time is requested in this model: b, p and d.
3. But the area under the whole curve depends on σ as well as on μ, as we see upon inserting (66) instead of P into (75), that is

$$\frac{1-erf\left(\frac{\sigma}{\sqrt{2}}\right)}{2}+\frac{e^{\frac{\sigma^2}{2}-\mu}}{\sqrt{2\pi}\,\sigma}\cdot\frac{2\,(d-p)}{3}=A\,(\mu,\sigma)\,. \tag{77}$$

4. Also p is to be replaced by its expression (65) in terms of σ and μ, yielding the new equation

$$\frac{1-erf\left(\frac{\sigma}{\sqrt{2}}\right)}{2}+\frac{e^{\frac{\sigma^2}{2}-\mu}}{\sqrt{2\pi}\,\sigma}\cdot\frac{2\left(d-b-e^{\mu-\sigma^2}\right)}{3}=. \\ =A\,(\mu,\sigma) \tag{78}$$

5. The meaning of (78) is that ***birth and death are fixed***, but the position of the ***peak may move*** according to the different numeric values of σ and μ.
6. In addition to that, we ***"dislike"*** the presence of the error function erf in (78) since this is not an "ordinary" function, i.e. it is one of the functions that mathematicians call "higher transcendental functions", having complicated formulae describing them. Thus, we would rather ***get rid of*** erf.
   How may we do so?

2.6. The area under the logpar curve depends on sigma only, and her is the area derivative W.R.T sigma

The simple answer to the last question 6) is "by differentiating both sides of (78) with respect to σ". In fact, the derivative of the erf function (70) is just the "Gaussian" exponential

$$\frac{d\,erf\,(x)}{dx}=\frac{2}{\sqrt{\pi}}\cdot e^{-x^2}. \tag{79}$$

and so the erf function itself ***will disappear*** by differentiating (78) with respect to σ. In fact, the derivative of the first term on the left hand side of (78) simply is, according to (79),

$$\frac{d}{d\sigma}\left[\frac{1-erf\left(\frac{\sigma}{\sqrt{2}}\right)}{2}\right]=-\frac{e^{-\frac{\sigma^2}{2}}}{\sqrt{2\pi}}. \tag{80}$$

As for the derivative with respect to σ of the second term on the left hand side of (78) we firstly notice that σ appears three times within that term. Thus, the relevant derivative is the sum of three terms, each of which includes the derivative of one of the three terms multiplied by the other two terms unchanged. In equations, one has:

$$\frac{d}{d\sigma}\left[\frac{1-erf\left(\frac{\sigma}{\sqrt{2}}\right)}{2}+\frac{e^{\frac{\sigma^2}{2}-\mu}}{\sqrt{2\pi}\,\sigma}\cdot\frac{2\left(d-b-e^{\mu-\sigma^2}\right)}{3}\right]= \\ =\frac{2^{\frac{3}{2}}e^{-\frac{\sigma^2}{2}}}{3\sqrt{\pi}}-\frac{e^{-\frac{\sigma^2}{2}}}{\sqrt{2\pi}}-\frac{\sqrt{2}\left(-e^{\mu-\sigma^2}+d-b\right)e^{\frac{\sigma^2}{2}-\mu}}{3\sqrt{\pi}\sigma^2}+ \\ +\frac{\sqrt{2}\left(-e^{\mu-\sigma^2}+d-b\right)e^{\frac{\sigma^2}{2}-\mu}}{3\sqrt{\pi}}. \tag{81}$$

Several alternative forms of this equation (81) are possible, and that is rather confusing. However, using a symbolic manipulator (this author did do by virtue of Maxima), a few steps lead to the following form of (81):

$$
\begin{aligned}
&\frac{dA\left(\mu\left(\sigma\right),\sigma\right)}{d\sigma} \equiv \frac{dA\left(\sigma\right)}{d\sigma} = \\
&= -\frac{\sqrt{2}\left(d-p\right)e^{-\frac{\sigma^2}{2}}}{3\sqrt{\pi}\left(p-b\right)\sigma^2} + \frac{\sqrt{2}\left(d-p\right)e^{-\frac{\sigma^2}{2}}}{3\sqrt{\pi}\left(p-b\right)} \\
&+ \frac{2^{\frac{3}{2}}e^{-\frac{\sigma^2}{2}}}{3\sqrt{\pi}} - \frac{e^{-\frac{\sigma^2}{2}}}{\sqrt{2\pi}} = \\
&= \frac{\left(p\,\sigma^2 - 2\,d\,\sigma^2 + b\,\sigma^2 - 2p + 2\,d\right)e^{-\frac{\sigma^2}{2}}}{3\sqrt{2\pi}\left(p-b\right)\sigma^2}.
\end{aligned}
\tag{82}
$$

This (82) is the derivative of the area with respect to sigma.

2.7. Exact "history equations" for each logpar curve

We now take a further, crucial step in our analysis of the logpar curve: we IMPOSE that the derivative of the area with respect to sigma, i.e. (82), is zero

$$\frac{dA\left(\sigma\right)}{d\sigma} = 0. \tag{83}$$

What does that mean?
Well, hold your breath: (82) is the Evo-SETI equivalent of the LEAST ACTION PRINCIPLE in physics ! This shocking conclusion does not show up at the moment, but it will at the end of this paper. For the time being with content ourselves with the "crude mathematics" of rewriting the imposed condition (83) by virtue of the last expression in (82) that, getting rid of both the exponential and the denominator, immediately boils down to

$$p\,\sigma^2 - 2\,d\,\sigma^2 + b\,\sigma^2 - 2p + 2\,d = 0. \tag{84}$$

This is just the quadratic equation in σ

$$\sigma^2\left(p - 2d + b\right) = 2\left(p - d\right) \tag{85}$$

and so we finally get

$$\sigma^2 = \frac{2\left(d-p\right)}{2\,d - \left(b+p\right)}. \tag{86}$$

This is the most important new result discovered in the present paper. It is the LOGPAR HISTORY EQUATION FOR σ

$$\sigma = \frac{\sqrt{2}\sqrt{d-p}}{\sqrt{2\,d - \left(b+p\right)}}. \tag{87}$$

In other words, given the input triplet (b, p, d) then (86) immediately yields the exact σ^2 of the b-lognormal left part of the logpar curve. It was discovered by this author on November 22, 2015, and led not only to this paper, but to the ***introduction of the ENERGY spent in a lifetime by a living creature, or by a whole civilization whose power-vs-time behaviour is given by the logpar curve***, as we will understand better in the coming sections of this paper.

At the moment, for reasons that will become obvious later, we confine ourselves to taking the limit of both sides of (87) for $d \to \infty$, with the result

$$\lim_{d\to\infty} \sigma = \lim_{d\to\infty} \frac{\sqrt{2}\sqrt{d-p}}{\sqrt{2\,d-(b+p)}} = \lim_{d\to\infty} \frac{\sqrt{2}\sqrt{d}}{\sqrt{2\,d}} = 1. \tag{88}$$

Since we already know that σ must be positive, (88) really shows that σ may range between zero and one only

$$0 \le \sigma \le 1 \tag{89}$$

Next to (87) one of course has a similar LOGPAR HISTORY EQUATION FOR μ, that is immediately derived from (65) and (87). To this end, just take the log of (65) to get

$$\mu = \ln(p-b) + \sigma^2 \tag{90}$$

that, invoking (86), yields the desired logpar equation for μ

$$\mu = \ln(p-b) + \frac{2\,(d-p)}{2\,d-(b+p)}. \tag{91}$$

In conclusion, our key two **LOGPAR HISTORY EQUATIONS** are

$$\begin{cases} \sigma = \dfrac{\sqrt{2}\sqrt{d-p}}{\sqrt{2\,d-(b+p)}} \\ \mu = \ln(p-b) + \dfrac{2\,(d-p)}{2\,d-(b+p)}. \end{cases} \tag{92}$$

2.8. Considerations on the logpar history equations

Some considerations on the logpar History Formulae (92) are now of order:

1. All these formulae are ***exact***, i.e. no Taylor series expansion was used to derive them.
2. But they were obtained by equalling to zero the ***derivative*** with respect to σ of the total area under the logpar curve given by (78).
3. Therefore the logpar History Formulae (92) are the equations of a ***minimum*** (we shall later show that this is indeed a minimum and not a maximum) ***of the*** $A(\sigma)$ ***function expressing the total area (78) as a function of*** σ.
4. One further question might be: μ and σ are independent variables in the Gaussian (and so in the lognormal, that is just e^,Gaussian). Since we differentiated (78) with respect to σ already, why dont we try differentiating it with respect to μ also? The answer is: because differentiating (78) with respect to μ leads to the ABSURD result $b = d$ i.e. one dies just when born! We leave the calculation to readers as an exercise, how funny!

2.9. Logpar peak coordinates expressed in terms of (b, p, d) only

Of particular importance for all future logpar applications is the expression of the peak coordinates (p, P) expressed in terms of the ***input triplet*** (b, p, d) only. Since the peak abscissa p is assumed to be known, we only have to derive the formula for the peak ordinate P. That is readily obtained by inserting the logpar History Formulae (92) into the peak height expression (66). After a few rearrangements, it is found to be given by

$$P = \frac{\sqrt{2\,d-(b+p)}}{2\sqrt{\pi}\sqrt{d-p}\,(p-b)} \cdot e^{-\frac{(d-p)}{2\,d-(b+p)}}. \tag{93}$$

2.10. History of Rome as an example of how to use the logpar history formulae

Let us go back to the History of Rome as summarized in the caption to Figure 4.

First of all, let us write down neatly the key three numeric input values in the History of Rome that were already mentioned in the caption to Figure 4:

$$\text{Rome\_input\_triplet} = \begin{cases} b = -753 \\ p = 117 \\ d = 476. \end{cases} \tag{94}$$

Then the logpar History Equations (92) immediately yield ***numerical values of the logpar σ and μ for Rome***, that we shall hereafter denote by σR and μR, respectively

$$\text{Rome\_logpar\_doublet} \begin{cases} \sigma R = 0.672 \\ \mu R = 7.221. \end{cases} \tag{95}$$

Next the study of the logpar ***peak*** comes. We already know from (94) that the abscissa of the peak of the Roman civilization was in 117 A.D. under Trajan

$$p_{Rome} = 117. \tag{96}$$

But the logpar peak ordinate for Rome must be found by virtue of (93). One thus gets

$$P_{Rome} = 5.44 \cdot 10^{-4} = 0.000544. \tag{97}$$

This is precisely the peak value shown in Figure 4.

2.11. The area under Rome's logpar and its meaning as overall energy of the roman civilization

Let us go back to (78), i.e. the total area under the logpar curve:

$$\frac{1 - erf\left(\frac{\sigma}{\sqrt{2}}\right)}{2} + \frac{e^{\frac{\sigma^2}{2}-\mu}}{\sqrt{2\pi}\,\sigma} \cdot \frac{2\left(d - b - e^{\mu-\sigma^2}\right)}{3} = A\left(\mu, \sigma\right). \tag{98}$$

If we insert the logpar History Formulae inside this equation, we obviously get the expression of the total area under the logpar as a function of just the input triplet (b, p, d) only. After some rearranging, this area formula turns out to be the rather complicated (but exact!) area equation:

$$\begin{aligned} &A\left(b, p, d\right) = \\ &= \frac{1 - erf\left(\frac{\sqrt{d-p}}{\sqrt{2\,d-(b+p)}}\right)}{2} \\ &+ \frac{\sqrt{d-p}\,\sqrt{2\,d-(b+p)}}{3\,\sqrt{\pi}\,(p-b)} \cdot e^{-\frac{d-p}{2\,d-(b+p)}}. \end{aligned} \tag{99}$$

What is the ***physical*** meaning of this area? If we consider the logpar curve as the curve of the ***power*** (measured in Watts) of the Roman civilization along the whole of its history course, then the area under this curve, i.e. the integral of the logpar between birth and death is the total energy (measured in Joules) spent by the civilization in its whole lifetime:

$$\begin{aligned} &\text{ENERGY\_spent\_in\_the\_Civilization\_LIFETIME} = \\ &= \int_b^d \text{POWER\_of\_that\_Civilization}\,(t)\; dt = \\ &= \int_b^d \text{logpar\_of\_that\_Civilization}\,(t)\; dt. \end{aligned} \tag{100}$$

In other words still, and in a much more general sense, if we know the power curve of any living being that lived in the past, like a cell, or an animal, or a human, or a Civilization of humans or of any other living forms (including ExtraTerrestrials), the integral of that power curve, i.e. logpar curve, between birth and death is the TOTAL ENERGY spent by that living form during the whole of its lifetime. Just an example regarding the last statement: if we assume that all Humans have potentially the ***same*** amount of energy to spend during their whole lifetime, then the logpar of great men who died young (like Mozart, for instance) must have the same area below their logpar and so a much higher peak since they lived shorter than others.
Let us go back to Rome's Civilization. Upon inserting the Rome's input triplet (94) into the area equation (99) the number is found

$$A_{\text{Rome}} = A\,(-753, 117, 476) = 0.381. \tag{101}$$

This is far from being equal to 1, the numeric value that would have made the Rome logpar curve to be a true ***probability density***. So, we have abandoned the use of probability densities (as all b-lognormals that this authors considered prior to 2017 were) but we have now our free hand to consider the ENERGY spent during a given lifetime. And the ENERGY is something profoundly different from the ENTROPY, that this author had previously considered (for instance with reference to his theorem that the Shannon Entropy of a Geometric Brownian Motion is a LINEAR function of the time, just as the MOLECULAR CLOCK is a LINEAR function of the time also, that is Kimuras theory of NEUTRAL evolution at the molecular level).

What a great step ahead we made: by releasing the normalization condition typical of probability densities, we were able to introduce ENERGY into the Evo-SETI theory. But does that mean that we have abandoned ENTROPY ? Not at all! Entropy and the Peak Locus Theorem supporting it, ARE STILL VALID in that the peak is the JUNCTION POINT BELONGING TO BOTH THE b-LOGNORMAL AND THE PARABOLA. Wow!

2.12. The energy function energy(d) as a function of the death instant d

Now we make a further "terrible step", since it's about death!

All of us would like to live as long as possible. The Conservation Instinct means just that. Then it makes sense to study the function (99) as a function of d only, meaning that we may increase the logpar area as much as life allows, i.e. as much as d keeps "going to the right" along the time axis. In fact, we are now NOT working with probability densities any more, and no normalization condition is blocking us any more. Consider thus the new function of d that we call ***energy*** (or, more correctly LifetimeEnergy)

$$\text{Energy}\,(d) = A\,(b, p, d) = \\ = \frac{1 - erf\left(\frac{\sqrt{d-p}}{\sqrt{2\,d-(b+p)}}\right)}{2} \\ + \frac{\sqrt{d-p}\,\sqrt{2\,d-(b+p)}}{3\,\sqrt{\pi}\,(p-b)} \cdot e^{-\frac{d-p}{2\,d-(b+p)}}. \tag{102}$$

What is the mathematics of this as a function of d?
First of all, (102) starts at the peak abscissa p because one's death cannot come earlier than one's peak p: more correctly, even in the worse cases of someone being "suddenly killed", we may always say that the peak of his life was the instant "just before his death", and so the mathematical definition

$$d \geq p \tag{103}$$

applies. Then, inserting $d = p$ into (102) and noticing from (70) that $erf(0) = 0$ we find

$$\text{Energy}\,(p) = \\ = \frac{1 - erf\left(\frac{\sqrt{0}}{\sqrt{2\,d-(b+p)}}\right)}{2} + 0 \cdot e^{-\,0} = \frac{1}{2}. \tag{104}$$

Why should the Energy of someone at its peak be equal to just $\frac{1}{2}$, and not to any other positive value (in Joules)? Well, worry not: we will solve this matter of extending the Energy at peak from $\frac{1}{2}$ to any other positive value in the next section of this paper. But, for the time being, please just content yourself of using the conventional value $\frac{1}{2}$. Thanks.

The next question is: what is the first derivative of (102) with respect to d? The answer is that, since d appears six times in (102), its derivative is the sum of six terms such that each term contains the derivative with respect to one precise d out of the six terms, and this derivative is multiplied by the other five terms unchanged. We did this calculation by virtue of the Maxima symbolic manipulator (web site: `http://maxima.sourceforge.net/`) and the result appears below as the equation in Eq. 105.

$$\frac{\sqrt{p-d}\,e^{\frac{d-p}{p-2d+b}}}{3\,\sqrt{\pi}(p-b)\,\sqrt{p-2d+b}} + \frac{\sqrt{p-2d+b}\,e^{\frac{d-p}{p-2d+b}}}{6\,\sqrt{\pi}(p-b)\,\sqrt{p-d}} \\ - \frac{\sqrt{p-2d+b}\,\sqrt{p-d}\left(\frac{1}{p-2d+b} + \frac{2(d-p)}{(p-2d+b)^2}\right)e^{\frac{d-p}{p-2d+b}}}{3\,\sqrt{\pi}(p-b)} \\ - \frac{\left(\frac{\sqrt{p-d}}{(p-2d+b)^{3/2}} - \frac{1}{2\,\sqrt{p-2d+b}\,\sqrt{p-d}}\right)e^{-\frac{p-d}{p-2d+b}}}{\sqrt{\pi}} \tag{105}$$

By equalling to zero the above equation, we may check whether the LifetimeEnergy (102) has any maxima or minima. The result given by Maxima is that, equalling to zero the above first derivative (Eq. 105), one gets the equation in the Eq. 106.

$$\frac{\sqrt{p-d}(p^2+(4b-6d)p+4d^2-2bd-b^2)e^{-\frac{p-d}{p-2d+b}}}{(3p^3-9dp^2+(6d^2+6bd-3b^2)p-6bd^2+3b^2d)} \cdot \frac{1}{\sqrt{\pi}\sqrt{p-2d+b}} = 0 \tag{106}$$

In turn, (Eq. 106) amounts to

$$p^2+(4b-6d)\,p+4d^2-2bd-b^2=0. \tag{107}$$

This is a quadratic equation in d that, solved for d, yields two roots. Discarding the one root having a minus sign in front, we finally are left with

$$d_{\text{abscissa\_of\_minimum\_Energy}} = \frac{\left(\sqrt{5}+3\right)p+\left(1-\sqrt{5}\right)b}{4}. \tag{108}$$

This is the abscissa of the minimum of the Energy: we could prove that it is really a minimum, rather than a maximum, by computing the second derivative of (102) with respect to d and then insert (108) there and show that the result is a purely positive number, but we shall not do so here for the sake of brevity.

2.13. Discovering an oblique asymptote of the energy function energy(d) while the death instant d is increasing indefinitely

This author discovered, around April 2016, that the Energy (102) has an ***oblique asymptote*** for $d \to \infty$.

Before we derive the equation of this oblique asymptote, however, a careful understanding of what d means is of order. We always said that the logpar theory described in this paper only necessitates the three inputs (b, p, d). But from this section onward we are going to consider ***higher and higher*** values of the death instant d so that the area under the logpar, i.e. the energy of the phenomenon of which the logpar is the power, may assume any assigned value. Thus, in the sections of this paper to follow, the death time d becomes a sort of ***new independent variable*** D rather than just one of the three fixed inputs (b, p, d). In other words, from now on, we will be careful to make the distinction between

1. The fixed, i.e. known, death instant d and
2. The movable, i.e. independent variable D allowing us to extrapolate into the future the logpar having the three fixed input values (b, p, d).

Having so said, the ENERGY (102) must more correctly be rewritten as a function of D rather than d

$$\text{Energy}(D) = \frac{1-erf\left(\frac{\sqrt{D-p}}{\sqrt{2D-(b+p)}}\right)}{2} + \frac{\sqrt{D-p}\sqrt{2D-(b+p)}}{3\sqrt{\pi}(p-b)} \cdot e^{-\frac{D-p}{2D-(b+p)}}. \tag{109}$$

Let now consider the definition of oblique asymptote given in elementary Calculus textbooks: if the limit

$$\lim_{D\to\infty} \left[Energy\,(D) - (m\,D + q)\right] \tag{110}$$

exists, then the Energy curve $Energy(D)$ approaches more and more the straight line

$$y_{\text{oblique\_asymptote}}\,(D) = m\,D + q. \tag{111}$$

Differentiating (111) with respect to D we immediately see that the angular coefficient m of the oblique asymptote is given by the limit for $D \to \infty$ of the first derivative of the energy (102), that we know to be given by the lengthy expression in Eq. 105. Maxima yielded:

$$m = \lim_{D\to\infty} \frac{dEnergy\,(D)}{dD} = \frac{\sqrt{2}}{3\,\sqrt{\pi}\,\sqrt{e}\,(p-b)}. \tag{112}$$

The same would of course been found had we considered the limit

$$\lim_{D\to\infty} \frac{Energy\,(D)}{D} = \lim_{D\to\infty} \frac{m\,D + q}{D} = m. \tag{113}$$

As for the asymptote's intercept with the vertical axis, q, (111) shows that it is given by the limit

$$q = \lim_{D\to\infty} \left[Energy\,(D) - m\,D\right]. \tag{114}$$

Thus, (103), (113) and Maxima yielded the result

$$q = \frac{1 - erf\left(\frac{1}{\sqrt{2}}\right)}{2} - \frac{b+p}{3\,\sqrt{2\pi e}\,(p-b)}. \tag{115}$$

Note: Maxima was unable to compute the limit (114) in a single shot: we had to do separately the two limits

$$\lim_{D\to\infty} \frac{1 - erf\left(\frac{\sqrt{D-p}}{\sqrt{2D-(b+p)}}\right)}{2} = \frac{1 - erf\left(\frac{1}{\sqrt{2}}\right)}{2} \tag{116}$$

and the $\infty - \infty$ limit (requiring L'Hospital's rule)

$$\begin{aligned}&\lim_{D\to\infty} \left[\frac{\sqrt{D-p}\,\sqrt{2\,D-(b+p)}}{3\,\sqrt{\pi}\,(p-b)} \cdot e^{-\frac{D-p}{2\,D-(b+p)}}\right.\\ &\left.-\frac{\sqrt{2}\,D}{3\,\sqrt{\pi}\,\sqrt{e}\,(p-b)}\right] = \frac{-(b+p)}{3\,\sqrt{2}\,\sqrt{\pi}\,\sqrt{e}\,(p-b)}.\end{aligned} \tag{117}$$

In conclusion, the oblique asymptote to the Energy (109) is given by

$$\begin{aligned}&y_{\text{oblique\_asymptote}}\,(D) = m\,D + q =\\ &= \frac{\sqrt{2}\,D}{3\,\sqrt{\pi}\,\sqrt{e}\,(p-b)} + \left[\frac{1 - erf\left(\frac{1}{\sqrt{2}}\right)}{2} - \right.\\ &\left.\frac{b+p}{3\,\sqrt{2\pi e}\,(p-b)}.\right] =\\ &= \frac{2\,D-(b+p)}{3\,\sqrt{2\pi e}\,(p-b)} + \frac{1 - erf\left(\frac{1}{\sqrt{2}}\right)}{2}.\end{aligned} \tag{118}$$

That is

$$y_{\text{oblique\_asymptote}}(D) = \frac{2\,D - (b+p)}{3\,\sqrt{2\pi e}\,(p-b)} + \frac{1 - erf\left(\frac{1}{\sqrt{2}}\right)}{2}. \tag{119}$$

2.14. The oblique asymptote for the history of rome case

A few comments about this oblique asymptote of the Energy (102) are now of order for the ”History of Rome” case:

1. Let us seek the year when the asymptote crosses the horizontal line having value $\frac{1}{2}$ as in Figure 5. In other words, let us look for the year after the Middle Ages, i.e. in the Renaissance, when the Energy acquired again the same numerical value that it had had in the Roman Empire at the time of Trajan. This means replacing the left-hand side of (119) by $\frac{1}{2}$ and then solving the resulting linear algebraic equation for D. One then gets

$$D = \frac{b+p}{2} + \frac{3\,\sqrt{2\pi e}\,(p-b)\,erf\left(\frac{1}{\sqrt{2}}\right)}{4}. \tag{120}$$

   Inserting the Rome values (94) into (120), we finally get the Renaissance year 1522.945 $\sim$ 1523 for the full recovery of the Energy back to the Trajan 117 A.D. value.
2. The numeric value $\frac{1}{2}$ for the horizontal line in Figure 5 is of course a ”remnant” of the way we derived things in all previous sections, and does not apply to the true numeric values of Ancient Rome. To find these true values, please read first the Wikipedia site `https://en.wikipedia.org/wiki/Roman_economy` a part of which we now repeat just the same here for convenience. ”Economic historians vary in their calculations of the gross domestic product (GDP) of the Roman economy during the Empire. In the sample years of 14, 100, and 150 A.D., estimates of per-capita GDP range from 166 to 380 sesterces (denoted HS). The GDP per capita of Italy is estimated as 40 to 66 percent higher than in the rest of the Empire, due to tax transfers from the provinces and the concentration of elite income in the heartland. In the ScheidelFriesen economic model, the total annual income generated by the Empire is placed at nearly 20 billion HS, with about 5 percent extracted by central and local government. Households in the top 1.5 percent of income distribution captured about 20 percent of income. Another 20 percent went to about 10 percent of the population who can be characterized as a non-elite middle. The remaining ”vast majority” produced more than half of the total income, but lived near subsistence. All cited economic historians stress the point that any estimate can only be regarded as a rough approximation to the realities of the ancient economy, given the general paucity of surviving pertinent data.”
   So, the above $\frac{1}{2}$ value must really be replaced by, say,

$$\frac{166+380}{2} 10^9 Hs = 273 \cdot 10^9 Hs = \text{273 billion Sestertii.} \tag{121}$$

3. The inclination of the oblique asymptote (119) is of course given by its angular coefficient (112) that we repeat here for convenience

$$m = \frac{\sqrt{2}}{3\,\sqrt{\pi}\,\sqrt{e}\,(p-b)} = \frac{0.1613138163461}{(p-b)}. \tag{122}$$

   This is inversely proportional to $(p-b)$, that is the initial part of the logpar when the Civilization grows up like as b-lognormal from birth to peak. If this $(p-b)$ is ”long” (like

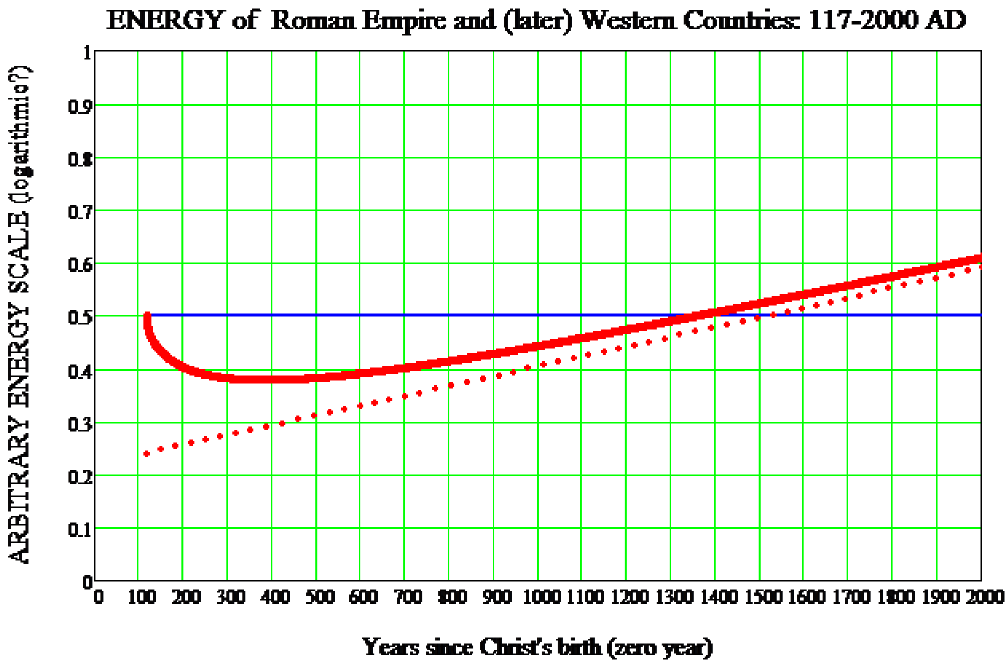

Fig. 5. Total ENERGY i.e. total WORK produced each year by the Roman Empire starting after its peak, that occurred in the year 117 under Emperor Trajan. This solid red curve is given by equation (109). We see that, after Trajan, the empire started to decline, producing less and less total energy and reaching its minimum in the year 385.844AD $\approx$ 386AD. These were the years (and actually decades, or even a few centuries) of the Barbaric Invasions inside the Western Roman Empire, after the Visigoths had inflicted the first severe defeat to the Romans at the battle of Hadrianople in 378 A.D.. Then, the Dark Ages of the Western Civilizations, or Middle Ages, lasted for about ten centuries, and it was not until about 1300 A.D. that Western Europe started flourishing again, reaching about the same Total Energy level that the former Roman Empire had under Trajan. This level is shown in the above graph by the thin solid blue horizontal line. After roughly 1300 A.D., the Italian Renaissance developed and then expanded into the whole of Western Europe in the following centuries. In addition, the dot-dot red line is the oblique asymptote to the Total Energy given by equation (119). Finally, while the horizotal time scale is in agreement with the historic facts, the vertical scale of this graph is completely ARBITRARY, and we shall RE-SCALE it to the correct Energy value (measured in Joules) in the coming sections of this paper.

Rome's, where one had

$$(p-b)_{\text{Rome}} = 117 - (-753) = 870 \tag{123}$$

years) then the oblique asymptote is "only slightly inclined", for Rome being

$$m_{\text{Rome}} = 1.8541817970815579 \cdot 10^{-4}. \tag{124}$$

In the limit for an "infinitely long growth", i.e. $(p-b) \to \infty$, the asymptote would be horizontal. On the contrary, for a "small" $(p-b)$ the asymptote would be "highly inclined", and just vertical for $(p-b)$=0.

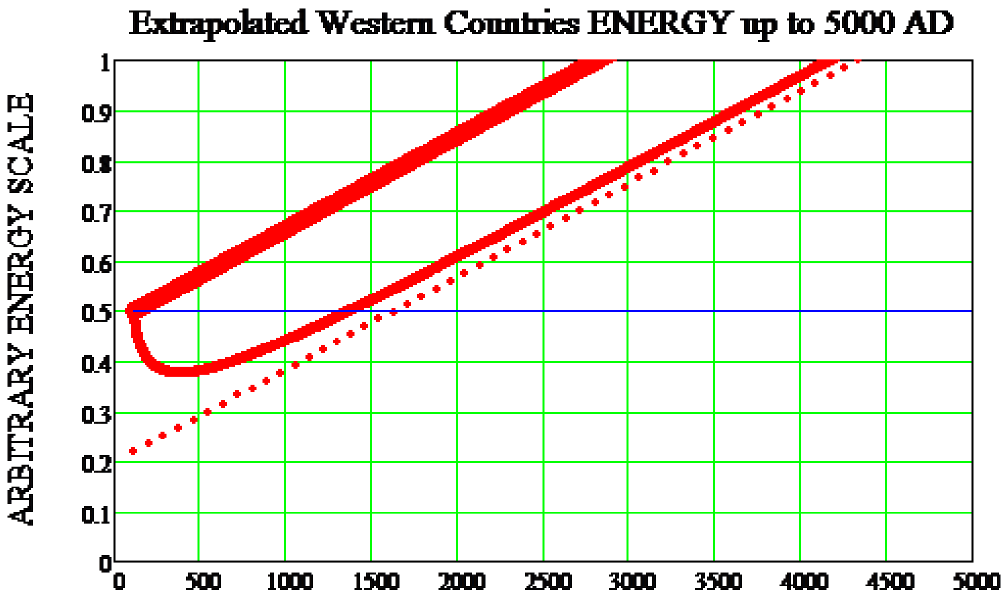

Fig. 6. NoSetbackEnergy (thick red upper straight line) i.e. Energy of the Western = Romance Countries (successors to the Roman Empire) had Rome not fallen. We see that it starts at the time of Trajan (117 A.D.) and keeps growing continuously ***parallel to the oblique asymptote of the true "concave" curve of the actual fall-and-recovery Energy.***

2.15. What if hadn't rome fallen? discovering the straight line parallel to the asymptote but starting at the rome power peak

What if hadn't Rome fallen?

This "silly question" isn't that silly, as it will let us discover the straight line ***parallel*** to the energy asymptote (119) but starting at the power peak (p, P).

Let us first understand the new problem we are solving in this section. Carl Sagan in his seminal book "Cosmos" (ref. Sagan (1980), see in particular the diagram on page 335) correctly described the Middle Ages (or "Dark Ages") "a poignant lost opportunity for the human Species that plagued the Western Civilization for about 1,000 years" since the fall of Rome in 476 A.D. up to the Italian Renaissance starting around 1300 A.D..

However, these lost 1,000 years become a "small time" if we consider the future development of the human civilization over periods of thousands of years if not even millions of years, which is what might already have happened to other ET Civilizations that our SETI astronomers are now looking for.

For more details and two plots, please see Figures 6 and 7.

In conclusion, we might say that:

1. **We only used three inputs (birth, peak, death) to define any LOGPAR power curve. The word "power" here is intended both in its physical sense (i.e. measured in Watts) and in the loose sense of "how much power" a certain civilization (or a certain physical phenomenon) displays along its whole lifetime from birth to death.**

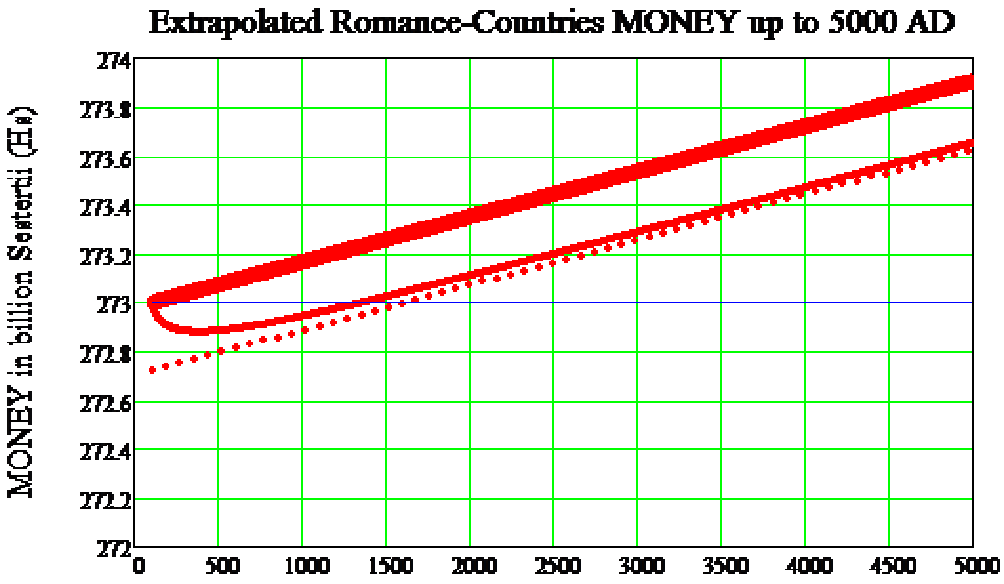

Fig. 7. Same as Figure 6 but with Energy replaced by MONEY in billion Sestertii (Sestertium is denoted HS). The inclination is obviously different with respect to Figure 6 because of the different scale on the vertical axis.

2. **The AREA under the LOGPAR power curve is the ENERGY that was produced during that lifetime. This is the striking, absolute novelty of this paper.**
3. **For all practical calculations, this ENERGY GROWS LINEARLY after the peak as given by equations (126) and its inverse (127) hereafter.**

Figure 6 shows the same $Energy(D)$ curve given by (109) but for a much longer time scale extrapolated into the future: the start is again at the year 0 (Christ born) but it now extends to the year 5,000 A.D.. Shown in Figure 6 is the same solid red Energy curve as in Figure 5 ***plus*** the very thick red straight line departing from the same "Trajan" point of coordinates $(p, Energy\_of\_Roman\_Empire\_under\_Trajan)$ and increasing constantly being the parallel straight line to the oblique asymptote (119). The meaning of Figure 6 is rather obvious: "in the long run" i.e. millennia after the "1000-years Dark Ages period", the total energy produced by the Western Countries is now "nearly" just the same is it would have been hadn't Rome fallen.

We shall call the Energy of this new straight line the "NoSetbackEnergy" (NSE). Its new equation is promptly derived:

1. Since the starting point has the abscissa p and the independent variable D starts at p (i.e. $D \geq p$), we shall simply multiply $(D - p)$ times the angular coefficient (112) of the **parallel** oblique asymptote.
2. The ordinate corresponding to the p abscissa is, by definition, given by

$$E(p) = \text{Roman\_Empire\_Energy\_in\_117\_AD}. \tag{125}$$

3. Then the equation of the $NoSetbackEnergy(D)$ is given by

$$NoSetbackEnergy\,(D) = \\ = E\,(D) = \frac{\sqrt{2}\,(D-p)}{3\,\sqrt{\pi}\,\sqrt{e}\,(p-b)} + E\,(p)\,. \tag{126}$$

4. Being linear in D, the last equation (126) has the advantage of being invertible, i.e. solvable for D

$$D = p + \frac{3\,\sqrt{\pi}\,\sqrt{e}\,(p-b)}{\sqrt{2}}\left[E\,(D) - E\,(p)\right]. \tag{127}$$

This allows one to find the ***future date*** D at which the $NoSetbackEnergy(D) = E(D)$ will reach any assigned value.
In conclusion, over long time scales, like the ones considered in the Evo-SETI Theory for Civilizations, whether human or Alien, the $NoSetbackEnergy(D) = E(D)$ is a simple and linear representation of how the Energy used (or produced) by that Civilization up to its peak will unroll in the times following the peak instant p, if $E(p)$ is known.

2.16. Mean power in a lifetime

In this section we are going to consider the notion of ***mean value*** of a logpar power curve.

Having abandoned the normalization condition for our logpar curves, clearly we may not use the same mean value definition of a random variable typical of probability theory.

However it's easy to use the **Mean Value Theorem for Integrals**. This is a variation of the **mean value theorem** which guarantees that a continuous function has at least one point where the function equals the average value of the function.

To translate the ***Mean Value Theorem for Integrals*** into a mathematical equation holding for logpar curves, we clearly have to start from the Area equation (99) with d replaced by D, and divide that area by the length of the $(D-b)$ segment in order to get the point along the vertical axis such that the area of the rectangle equals the Area (99). This is the required ***Mean Power Value over a lifetime*** and is given by

$$\text{Mean\_Power\_over\_a\_lifetime} = \frac{A\,(b,p,D)}{D-b} = \\ = \frac{\frac{1-erf\left(\frac{\sqrt{D-p}}{\sqrt{2\,D-(b+p)}}\right)}{2} + \frac{\sqrt{D-p}\,\sqrt{2\,D-(b+p)}}{3\,\sqrt{\pi}(p-b)}\cdot e^{-\frac{D-p}{2\,D-(b+p)}}}{D-b}. \tag{128}$$

It is interesting to consider the limit of the Mean Power over a lifetime (128) for $D \to \infty$. The calculation implies the use of L'Hospital's rule, and the result is

$$\text{Asymptotic\_Mean\_Power\_over\_a\_lifetime} = \\ = \lim_{D\to\infty}\left(\text{Mean\_Power\_over\_a\_lifetime}\right) = \\ = \frac{\sqrt{2}}{3\,\sqrt{\pi}\,\sqrt{e}\,(p-b)}. \tag{129}$$

Notice that this equation (129) is just the same as the derivative of the asymptotic energy equation (119) with respect to D. This must indeed be the case, since the Power in general is just the derivative of the Energy with respect to the time variable, which is D in this case.

By this we have completed the study of the mean along the vertical axis, i.e. the power axis. However, one might still wish to find, in some sense, "the mean value of what lies on the horizontal axis", i.e. the lifetime mean value. That is done in the next section.

2.17. Lifetime mean value

It is natural to seek for some mathematical expression yielding the mean value of a lifetime, meaning the mean value ***along the time axis*** of the $(D-b)$ time segment representing the ***lifetime*** of a living organism, or a civilization or even an ET civilization. We propose the following definition of such a lifetime mean value:

$$\begin{aligned} &\text{lifetime\_mean\_value} = \\ &= \int_b^p t \cdot b\_\text{lognormal}(t;\mu,\sigma,b)\ dt + \\ &\int_p^D t \cdot \text{parabola}(t)\ dt = \end{aligned} \tag{130}$$

inserting the b-lognormal (64) and the parabola (62) into (130), the latter is turned into

$$= \int_b^p t \cdot \frac{e^{-\frac{(\log(t-b)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma\ (t-b)} dt + \int_p^D t \cdot P\left[1 - \frac{(t-p)^2}{(D-p)^2}\right] dt. \tag{131}$$

The first integral may be computed in terms of the error function $erf(x)$ given by (70), and the result is

$$\begin{aligned} &\int_b^p t \cdot \frac{e^{-\frac{(\log(t-b)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma\ (t-b)} dt = \\ &= \frac{e^{\frac{\sigma^2}{2}+\mu}\left[1 - erf\left(\frac{\sigma^2 - \log(p-b)+\mu}{\sqrt{2}\,\sigma}\right)\right]}{2} + \\ &+ \frac{b\left[1 - erf\left(\frac{\log(p-b)-\mu}{\sqrt{2}\,\sigma}\right)\right]}{2} = \end{aligned} \tag{132}$$

that may be further simplified by resorting to (65), with the result

$$\begin{aligned} &\int_b^p t \cdot \frac{e^{-\frac{(\log(t-b)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma\ (t-b)} dt = \\ &= \frac{e^{\frac{\sigma^2}{2}+\mu}\left[1 - erf\left(\sqrt{2}\,\sigma\right)\right]}{2} + \frac{b\left[1 - erf\left(\frac{\sigma}{\sqrt{2}}\right)\right]}{2}. \end{aligned} \tag{133}$$

Re-expressing now (133) in terms of the History Formulae (92), it finally takes the form

$$\int_b^p t \cdot \frac{e^{-\frac{(\log(t-b)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma\,(t-b)} dt = \\ = \frac{(p-b)\; e^{\frac{3(D-p)}{2D-(b+p)}} \left[1 - erf\left(\frac{2\sqrt{D-p}}{\sqrt{2D-(b+p)}}\right)\right]}{2} + \\ + \frac{b\left[1 - erf\left(\frac{\sqrt{D-p}}{\sqrt{2D-(b+p)}}\right)\right]}{2}. \tag{134}$$

As for the second integral in (131), i.e. the parabola integral, it is promptly computed as follows

$$\int_p^D t \cdot P\left[1 - \frac{(t-p)^2}{(D-p)^2}\right] dt = \frac{P\,(D-p)\,(3D+5p)}{12}. \tag{135}$$

Inserting for P its expression (93), after some rearranging we conclude that the parabola integral is given by

$$\int_p^D t \cdot P\left[1 - \frac{(t-p)^2}{(D-p)^2}\right] dt = \\ = \frac{\sqrt{2\,D-(b+p)}\,\sqrt{D-p}\,(3D+5p)}{24\,\sqrt{\pi}\,(p-b)} \cdot \\ e^{-\frac{(D-p)}{2\,D-(b+p)}}. \tag{136}$$

In conclusion, the mean lifetime is found by summing (134) and (136) and reads

$$\text{lifetime\_mean\_value} = \\ = \frac{(p-b)\; e^{\frac{3(D-p)}{2D-(b+p)}} \left[1 - erf\left(\frac{2\sqrt{D-p}}{\sqrt{2D-(b+p)}}\right)\right]}{2} + \\ + \frac{b\left[1 - erf\left(\frac{\sqrt{D-p}}{\sqrt{2D-(b+p)}}\right)\right]}{2} + \\ + \frac{\sqrt{2\,D-(b+p)}\,\sqrt{D-p}\,(3D+5p)}{24\,\sqrt{\pi}\,(p-b)} \cdot \\ e^{-\frac{(D-p)}{2\,D-(b+p)}}. \tag{137}$$

Just to give a numerical example, let us find the mean lifetime of the Civilization of Rome. The first integral (134), by virtue of the Rome input triplet (94), yields the numeric value of the mean b-lognormal, i.e.

$$\text{mean\_value\_of\_Rome\_b-lognormal} = -35.599. \tag{138}$$

This means four years before the battle of Actium `https://en.wikipedia.org/wiki/Battle_of_Actium`, fought on 2 September 31 B.C.: a crucial event that saw Cleopatras Egypt being absorbed as just one more Roman province. On the other hand, the parabola integral (136) and the Rome input triplet (94) yield for the parabola mean value the year

$$\text{Rome\_parabola\_mean\_value} = 32.758. \tag{139}$$

This 33 A.D. was a year falling during the empire of Tiberius (14-37 A.D.), and, most important, is just the year when Jesus Christ was crucified in Jerusalem. So, by summing up the two equations (138) and (139),we reach the important conclusion that the mean value of the overall Rome logpar power curve is just -2.840, i.e.

$$\begin{aligned} &\text{mean\_value\_of\_Rome\_LOGPAR\_History} = \\ &= -2.840 \approx 3 \text{ B.C.} \end{aligned} \tag{140}$$

This is an outstanding result. Our Evo-SETI Theory, in the LOGPAR form described in this paper, predicts that the most important year in the History of Rome was just (or nearly to) the birth of Jesus Christ! With all due respect to religion, our "popular" comment to such a result would be... Jeee !

2.18. Logpar power curves vs. b-lognormal probability densities

Twenty months (November 2015 - July 2017) were necessary to this author to discover the logpar power curves and their properties as described in the present paper. In fact, prior to 2017, this author had built his Evo-SETI Theory on the notion of b-lognormal probability densities ***only***.

But now, the advent of logpar power curves enables a profound study of energy within the Evo-SETI Theory that previously was not allowed.

Energy means here both the energy displayed by a Civilization in time (like the Roman one, as exemplified in this paper) as well as the energy produced by a star (like the Sun). All this was impossible to do in the b-lognormal approach to Evo-SETI Theory studied by this author in the years prior to 2017, i.e. in roughly in the years 2011-2015.

Then, what about the Shannon Entropy as the measure of evolution intended as ever increasing amount of information (in bits) and complexity?

Well, ***the Peak-Locus Theorem proved around 2013-2014 by this author even for a generic mean value curve other than just the simple exponential mean curve (typical of Geometric Brownian Motion, GBM) is STILL VALID in the logpar Evo-SETI Theory.***

In fact, the b-lognormal part of the logpar (between birth and peak) remains even in the logpar approach. And so the proof remains of the key result asserting that the existence of the Molecular Clock (the fundamental discovery of molecular genetics made over 50 years ago) is derived in Evo-SETI Theory as a mathematical consequence of the Theory itself, just as Kepler's laws are derived in Newtonian mechanics as a mathematical consequence of Newton's Law of Gravitation. This is possible because ***the peak point is the junction point*** belonging to both the preceding b-lognormal (from which the Peak Locus Theorem and the Molecular Clock are derived) and to the ensuing parabola (from which the energy-related theorems are derived).

A summary of the advantages of the logpar power curve approach over the b-lognormal probability density approach is given in Table 1.

3. Conclusions about energy as a part of evo-seti theory

More and more exoplanets are now being discovered by astronomers either by observations from the ground or by virtue of space missions, like "CoRot", "Kepler", "Gaia", and other future space missions. As a consequence, a recent estimate sets at 40 billion the number of Earth-sized planets orbiting in the habitable zones of sun-like stars and red dwarf stars within the

Table 1. A summary of the advantages of the logpar power curve approach over the b-lognormal probability density approach

	b-lognormal **PROBABILITY DENSITY in the time** (used by this author 2012-2017)	**LOGPAR power curve in the time** (used by this author since 2017)
3 numeric INPUTS	birth b, senility (=descending inflexion) s, death d (tangent at s intercept with time axis). **DISADVANTAGE: estimating** s **is hard**.	birth b, peak p, death d (parabola intercept with the time axis). **ADVANTAGE:** b, p, d **are EASILY found**.
Normalization Condition in terms of the only truly independent variable σ **(**μ **is a FALSE independent variable)**	**Total Area under b-lognormal equals ONE** $\int_b^\infty \frac{e^{-\frac{[\ln(t-b)-\mu]^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma\,(t-b)}\,dt = 1$	**Total Area under the LOGPAR curve is** A $\frac{1-erf\left(\frac{\sigma}{\sqrt{2}}\right)}{2} + \cdot\, \frac{e^{\frac{\sigma^2}{2}-\mu}}{\sqrt{2\pi}\,\sigma} \cdot \frac{2\left(d-b-e^{\mu-\sigma^2}\right)}{3} = A$
Normalization Condition in terms of the three input parameters	**Given the three inputs** (b, s, d) **one has** $\int_b^\infty \frac{e^{-\frac{[\ln(t-b)-\mu]^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma\,(t-b)}\,dt = 1$	**Given the three inputs** b, p, d **one has** $\text{Energy}\,(d) = A\,(b, p, d) = \frac{1 - erf\left(\frac{\sqrt{d-p}}{\sqrt{2\,d-(b+p)}}\right)}{2} + \frac{\sqrt{d-p}\,\sqrt{2\,d-(b+p)}}{3\,\sqrt{\pi}\,(p-b)} \cdot e^{-\frac{d-p}{2\,d-(b+p)}}.$
History Formulae	$\begin{cases} \sigma = \frac{d-s}{\sqrt{d-b}\,\sqrt{s-b}} \\ \mu = \ln(s-b) + \frac{(d-s)(b+d-2s)}{(d-b)\,(s-b)}. \end{cases}$	$\begin{cases} \sigma = \frac{\sqrt{2}\,\sqrt{d-p}}{\sqrt{2\,d-(b+p)}} \\ \mu = \ln(p-b) + \frac{2\,(d-p)}{2\,d-(b+p)}. \end{cases}$
Mean value	Mean Value Theorem for the lognormal stochastic Process $L(t)$ starting at the initial time ts $m_L(t) \equiv \int_0^\infty n \cdot \frac{e^{-\frac{\left[\ln(n)-M_L(t)\right]^2}{2\sigma_L^2(t-ts)}}}{\sqrt{2\pi}\,\sigma_L\,\sqrt{t-ts}\cdot n}\,dn =$ $e^{M_L(t)}\,e^{\frac{\sigma_L^2}{2}(t-ts)}$ with its inverse $M_L(t) = \ln(m_L(t)) - \frac{\sigma_L^2}{2}(t-ts).$	LOGPAR lifetime_mean_value = $= \frac{(p-b)\;e^{\frac{3(D-p)}{2D-(b+p)}}\left[1-erf\left(\frac{2\sqrt{D-p}}{\sqrt{2D-(b+p)}}\right)\right]}{2} + \frac{b\left[1-erf\left(\frac{\sqrt{D-p}}{\sqrt{2D-(b+p)}}\right)\right]}{2} + \frac{\sqrt{2\,D-(b+p)}\,\sqrt{D-p}\,(3D+5p)}{24\,\sqrt{\pi}\,(p-b)} \cdot e^{-\frac{(D-p)}{2\,D-(b+p)}}.$
Asymptotic ENERGY for high values of Death D	———	$y_{\text{oblique\_asymptote}}(D) = \frac{2\,D-(b+p)}{3\,\sqrt{2\pi e}(p-b)} + \frac{1-erf\left(\frac{1}{\sqrt{2}}\right)}{2}.$

Milky Way galaxy. With such huge numbers of "possible Earths" in sight, Astrobiology and SETI are becoming research fields more and more attractive to a number of young scientists. Mathematically innovative papers like the Evo-SETI ones, revealing unsuspected relationships like the one between the Molecular Clock and the Entropy of b-lognormals in Evo-SETI Theory, should thus be welcome.

But in this paper we also showed how ENERGY can be added to ENTROPY in Evo-SETI Theory.

While just preserving all the advantages of the b-lognormal probability density functions, we kept these b-lognormals good only for the first part of the curve: the one between birth and peak. The second part, between peak and death, was replaced by just a simple descending half-parabola, thus avoiding any inflexion point like the ”senility” point typical of b-lognormals that was so difficult to estimate numerically in most cases. Thus LOGPAR curves have greatly simplified the description of any finite phenomenon in time like the lifetime of a cell, or a human, or a civilization (like the Rome one used in this paper as an example) or even like an ET civilization.

In addition to all this, we abandoned the normalization condition of b-lognormals retaining just their shape, and not the numbers. This transformed the logpars into power curves, both in the popular sense where ”power” means ”political & military power” and in the strictly physical sense, where ”power” means a curve measured in Watts. And the area under such a logpar is indeed the ENERGY associated to the logpar phenomenon between birth and death. So, for the first time in the creation of our Evo-SETI Theory, we were able to add ENERGY to the ENTROPY previously considered already. And energy and entropy are the two pillars of classical Thermodynamics thus making Evo-SETI even more neatly applicable to stars.

Finally, there is one more crucial step ahead that we made by introducing logpars. We actually ”stumbled” into the PRINCIPLE OF LEAST ENERGY. But an adequate description of that result would require more papers giving more profound justifications. So we stop at this point, asking our gentle readers to prepare for more mathematical papers in Evo-SETI Theory.

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