Mem. S.A.It. Vol. 85, 44

Memorie della

Standard and exotic singularities regularized by varying constants

M.P. Dąbrowski[1,2], K. Marosek[1,3], and A. Balcerzak[1,2]

[1] Institute of Physics, University of Szczecin, Wielkopolska 15, 70-451 Szczecin, Poland
e-mail: mpdabfz@wmf.univ.szczecin.pl
[2] Copernicus Center for Interdisciplinary Studies, Sławkowska 17, 31-016 Kraków, Poland
[3] Chair of Physics, Maritime University, Wały Chrobrego 1-2, 70-500 Szczecin, Poland

Abstract. We review the variety of new singularities in homogeneous and isotropic FRW cosmology which differ from standard Big-Bang and Big-Crunch singularities and suggest how the nature of these singularities can be influenced by the varying fundamental constants.

Key words. Cosmology: singularities – Cosmology: varying constants

1. Introduction

Currently, one is able to differentiate quite a number of cosmological singularities with completely different properties from a Big-Bang or a Big-Crunch. Many of them do not exhibit geodesic incompleteness, but they still lead to a blow-up of various physical quantities (scale factor, mass density, pressure, physical fields). In this paper we will discuss how they can be influenced by the variability of the fundamental constants.

2. Standard and exotic singularities in cosmology.

Standard Einstein-Friedmann equations are two equations for three unknown functions of time $a(t), p(t), \varrho(t)$ - the scale factor, the pressure, the mass density. In order to solve them, usually the equation of state is specified. Most common form of it is a barotropic one $p = w\varrho c^2$ with w being a barotropic index, c velocity of light. However, it is interesting that one obtains the independent evolution of the mass density and pressure provided we do not assume any equation of state which tights these quantities.

Until quite recently, including first supernovae results (Perlmutter et al. 1999), most cosmologists studied only the simplest - say "standard" solutions of the Friedmann equation. They each begin with a Big-Bang (BB) singularity for which $a \to 0, \varrho, p \to \infty$, while in the future one of them (of positive curvature $k = +1$) terminates at a second singularity (Big-Crunch - BC), where $a \to 0, \varrho, p \to \infty$ and the other two ($K = 0, -1$) continue to an asymptotic emptiness $\varrho, p \to 0$ for $a \to \infty$. BB and BC exhibit geodesic incompletness and a curvature blow-up. In fact, the first supernovae observations gave evidence for the strong energy condition (SEC) ($\varrho c^2 + 3p \geq 0$, $\varrho c^2 + p \geq 0$) violation, but the paradigm of

Send offprint requests to: M.P. Dąbrowski the "standard" Big-Bang/Crunch singularities remained untouched.

However, a combined bound on the dark energy barotropic index w (Tegmark 2004) showed that there was no sharp cut-off of the data at $p = -\varrho$ so that the dark energy with $p < -\varrho$ (phantom), could also be admitted. This lead to the cosmic "no-hair" theorem violation since even a small fraction of phantom dark energy could dominate the evolution instead of the cosmological term. Since $w < -1$ for phantom, then we may define $\mid w + 1 \mid = -(w + 1) > 0$, so $a(t) = t^{-2/3|w+1|}$ and the conservation law gives $\varrho \propto a^{3|w+1|}$. This means that if the universe grows bigger, its density is higher, and finally it becomes dominated by phantom dark energy. An exotic future singularity – a Big-Rip (BR) – appears, for which $\varrho, p \to \infty$ for $a \to \infty$ (Caldwell 2002). At Big-Rip the null energy condition (NEC), the weak energy condition (WEC), and the dominant energy condition (DEC) are all violated. Also, the curvature invariants R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverge in the same way as at BB and at BC, and there is a geodesic incompleteness at Big-Rip as well. Besides, everything is pulled apart on the approach to such a singularity in a reverse order.

Observational support for a Big-Rip gave an inspiration for studies of other exotic types of singularities as possible sources of dark energy. Especially, Barrow (2004a) showed that if one drops an assumption about the imposition of the equation of state, and specifies the scale factor as

$$a(t) = a_s \left[\delta + (1-\delta)\, y^m - \delta\, (1-y)^n\right] \;\; y \equiv \frac{t}{t_s} \, , \qquad (1)$$

where $a_s \equiv a(t_s) =$ const. and $\delta, m, n =$ const., then one gets (apart from a Big-Bang at $t = 0$) a new type of singularity at $t = t_s$ (provided $1 < n < 2$) which was christened a Sudden Future Singularity (SFS). Such a singularity is a singularity of pressure p (or $\ddot{a}$) and leads merely to DEC violation. The standard "Friedmann limit" is easily obtained by taking the "nonstandardicity" parameter $\delta \to 0$ in (1). In fact, at SFS we have:

$$a = \text{const.}, \quad \dot{a} = \text{const.} \quad \varrho = \text{const.}$$
$$\ddot{a} \to -\infty \quad p \to \infty \quad \text{for} \quad t \to t_s \, . \qquad (2)$$

It is interesting that the Schwarzschild horizon at $r = r_g$ has a singular metric, while the curvature invariants are regular there. On the ohter hand, an SFS at $t = t_s$ has a regular metric, while curvature invariants diverge.

The matter related to SFS may serve as dark energy, especially if they are quite close in the near future. For example, an SFS may even appear in 8.7 Myr with no contradiction to a bare supernovae data. It can be fitted to a combined SnIa, CMB and BAO data, but at the expense of admitting on the approach to a Big-Bang a fluid which is not exactly dust (m=0.66), but has slightly negative pressure (m = 0.73, w = -0.09) (Denkiewicz et al. 2012). A more general class of singularities known as Generalized Sudden Future Singularities (GSFS) which do not violate any of the energy conditions are also possible (Barrow 2004b).

There is yet the whole class of non-Big-Bang singularities (Nojiri et al. 2005; Dąbrowski & Denkiewicz 2010) (Finite Scale Factor, Big Separation, w-singularities (Dąbrowski & Denkiewicz 2009), Little-Rip (Frampton et al. 2011), Pseudo-Rip (Frampton et al. 2012) and their various versions like Big-Boost and Big-Brake (belonging to an SFS class Gorini et al. 2004), Big-Freeze (belonging to an FSF class Bouhmadi-Lopez et al. 2008), generalized Big-Separation (Yurov 2010) and generalized w-singularities (Yurov 2010). Most of them can be described using one unified scale factor (Dąbrowski & Marosek 2013) reading as

$$a(t) = a_s \left(\frac{t}{t_s}\right)^m \exp\left(1 - \frac{t}{t_s}\right)^n , \qquad (3)$$

with the appropriate choice of constants t_s, a_s, m, n. In fact, from (3) we can see that for $0 < m < 2/3$ we deal with a BB singularity and $a \to 0$, $\varrho \to \infty$, $p \to \infty$ at $t \to 0$. For $m < 0$ we have a BR singularity with $a \to \infty$, $\varrho \to \infty$, $p \to \infty$ at $t = 0$. An SFS appears for $1 < n < 2$ at $t = t_s$ ($a = a_s$, $\varrho =$ const., $p \to \infty$), and an FSF appears for $0 < n < 1$ at $t = t_s$ ($a = a_s$, $\varrho \to \infty$, $p \to \infty$).

In order to classify the strength of standard and exotic singularities some definitions have been proposed. According to Tipler (1977) a singularity is weak if a double integral

$$\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' R_{ab}u^a u^b \tag{4}$$

does not diverge on the approach to a singularity at $\tau = \tau_s$ (τ is the proper time), while according to Królak (1988) a singularity is weak if a single integral

$$\int_0^{\tau} d\tau' R_{ab}u^a u^b \tag{5}$$

does not diverge on the approach to a singularity at $\tau = \tau_s$. Otherwise, a singularity is strong. From now on, T will stand for the definition of Tipler, and K will stand for the definition of Królak. It is interesting that both point particles and even extended objects may not feel weak singularities and can pass through them (Fernandez-Jambrina & Lazkoz 2006; Balcerzak & Dąbrowski 2006).

Classification of exotic singularities was first given by Nojiri et al. (2005) and further developed by Dąbrowski & Denkiewicz (2010). Their current classification is attempted in Table 1.

3. Varying constants theories.

First fully quantitative framework which allowed for variability of the fundamental constant was Brans-Dicke scalar-tensor gravity. The gravitational constant G in such a theory is associated with an average gravitational potential (scalar field) Φ surrounding a given particle: $< \Phi >= GM/(c/H_0) \propto 1/G = 1.35 \times 10^{28} g/cm$. The scalar field Φ gives the strength of gravity

$$G = \frac{1}{16\pi\Phi} , \tag{6}$$

which changes Einstein-Hilbert action into Brans-Dicke action

$$S = \int d^4x \sqrt{-g}\left(\Phi R - \frac{\omega}{\Phi}\partial_\mu\Phi\partial^\mu\Phi + L_m\right) , \tag{7}$$

and further relates to low-energy-effective superstring action when $\omega = -1$. In superstring theory, the string coupling constant $g_s = \exp(\phi/2)$ changes in time with ϕ being the dilaton, and $\Phi = \exp(-\phi)$.

Another framework is given by varying speed of light theories (VSL). In Albrecht & Magueijo (1999) model (AM) the speed of light is replaced by a scalar field

$$c^4 = \psi(x^\mu) , \tag{8}$$

leading to the action

$$S = \int d^4x \sqrt{-g}\left[\frac{\psi R}{16\pi G} + L_m + L_\psi\right] . \tag{9}$$

AM model breaks Lorentz invariance (relativity principle and light principle). There is a preferred frame (called a cosmological or a CMB frame) in which the field is minimally coupled to gravity. This model solves basic problems of standard cosmology such as the horizon problem and the flatness problem. One of the ansätze is that $\rho = \rho_0 a^{-3(w+1)}$, $c(t) = c_0 a^n$ which solves the above two problems if $n \leq -(1/2)(3w+1)$ (Albrecht & Magueijo 1999). Another version of VSL is Magueijo covariant (conformally) and locally invariant model (Magueijo 2000):

$$\psi = \ln\left(\frac{c}{c_0}\right) \quad \text{or} \quad c = c_0 e^{\psi} , \tag{10}$$

with the action

$$S = \int d^4x \sqrt{-g}\left[\frac{c_0^4 e^{\alpha\psi}(R + L_\psi)}{16\pi G} + e^{\beta\psi}L_m\right] , \tag{11}$$

and

$$L_\psi = \kappa(\psi)\nabla_\mu\psi\nabla^\mu\psi . \tag{12}$$

There is an extra condition that $\alpha - \beta = 4$ with interesting subcases: a) $\alpha = 4; \beta = 0$, giving Brans-Dicke theory with $\phi_{BD} = e^{4\psi}/G$ and $\kappa(\psi) = 16\omega_{BD}(\phi_{BD})$; b) $\alpha = 0; \beta = -4$, called a minimal VSL theory.

Yet another framework is varying fine structure constant α theory (or varying charge $e = e_0\epsilon(x^\mu)$ theory (Webb et al. 1999)

$$\begin{aligned} S &= \int d^4x \sqrt{-g}\left(\psi R - \frac{\omega}{2}\partial_\mu\psi\partial^\mu\psi\right. \\ &- \left.\frac{1}{4}f_{\mu\nu}f^{\mu\nu}e^{-2\psi} + L_m\right) \end{aligned} \tag{13}$$

Table 1. Classification of singularities in FRW cosmology

Type	Name	t sing.	a(t$_s$)	$\varrho(t_s)$	p(t$_s$)	$\dot{p}(t_s)$ etc.	w(t$_s$)	T	K
0	Big-Bang (BB)	0	0	∞	∞	∞	finite	strong	strong
I	Big-Rip (BR)	t_s	∞	∞	∞	∞	finite	strong	strong
I$_l$	Little-Rip (LR)	∞	∞	∞	∞	∞	finite	strong	strong
I$_p$	Pseudo-Rip (PR)	∞	∞	finite	finite	finite	finite	weak	weak
II	Sudden Future (SFS)	t_s	a_s	ϱ_s	∞	∞	finite	weak	weak
II$_g$	Gen. Sudden Future (GSFS)	t_s	a_s	ϱ_s	p_s	∞	finite	weak	weak
III	Finite Scale Factor (FSF)	t_s	a_s	∞	∞	∞	finite	weak	strong
IV	Big-Separation (BS)	t_s	a_s	0	0	∞	∞	weak	weak
V	w-singularity (w)	t_s	a_s	0	0	0	∞	weak	weak

in which the scalar field is associated with electric charge $\psi = \ln\epsilon$ and $f_{\mu\nu} = \epsilon F_{\mu\nu}$. This model can be related to the VSL theories due to the definition of the fine structure constant $\alpha(t) = e^2/[\hbar c(t)]$. If one assumes the linear expansion of $e^{\psi} = 1 - 8\pi G\zeta(\psi - \psi_0) = 1 - \Delta\alpha/\alpha$ with the constraint on the local equivalence principle violence $|\,\zeta\,| \leq 10^{-3}$, then the relation to dark energy density parameter Ω_ψ is

$$w + 1 = \frac{(8\pi G \frac{d\psi}{d\ln a})^2}{\Omega_\psi}\,, \tag{14}$$

which can be tested (while mimicking the dark energy) by spectrograph CODEX (COsmic Dynamics EXplorer) – a device attached to a planned E-ELT (European Extremely Large Telescope) measuring the redshift drift effect (or Sandage-Loeb effect - Sandage 1962; Loeb 1998) for $2 < z < 5$ (Vielzeuf & Martins 2012).

4. Varying constant versus cosmic singularities.

It has been shown that quantum effects (Houndjo 2010) may change the strength of exotic singularities (e.g. an SFS to become an FSF). As it was already mentioned, varying constants cosmologies have been applied to solve standard cosmology problems as well. Our idea is to apply them to solve the singularity problem in cosmology. We can also ask if varying constants theories can soften/strengthen the standard and exotic singularities?

We consider the Friedmann universes in varying speed of light (VSL) theories and varying gravitational constant G theories as follows (ϱ - mass density; $\varepsilon = \varrho c^2(t)$ - energy density in $Jm^{-3} = Nm^{-2} = kgm^{-1}s^{-2}$)

$$\varrho(t) = \frac{3}{8\pi G(t)}\left(\frac{\dot{a}^2}{a^2} + \frac{kc^2(t)}{a^2}\right), \tag{15}$$

$$p(t) = -\frac{c^2(t)}{8\pi G(t)}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2(t)}{a^2}\right), \tag{16}$$

with the source terms in the energy-momentum "conservation law" due to varying c and G:

$$\dot{\varrho}(t) + 3\frac{\dot{a}}{a}\left(\varrho(t) + \frac{p(t)}{c^2(t)}\right) = -\varrho(t)\frac{\dot{G}(t)}{G(t)} + 3\frac{kc(t)\dot{c}(t)}{4\pi G a^2}\,.$$

For flat $k = 0$ universes we have

$$\varrho(t) = \frac{3}{8\pi G(t)}\left[\frac{m}{t} - \frac{n}{t_s}\left(1 - \frac{t}{t_s}\right)^{n-1}\right]^2, \tag{17}$$

$$\begin{aligned} p(t) = & -\frac{c^2(t)}{8\pi G(t)}\left[\frac{m(3m-2)}{t^2} - 6\frac{mn}{tt_s}\left(1 - \frac{t}{t_s}\right)^{n-1}\right. \\ & + 3\frac{n^2}{t_s^2}\left(1 - \frac{t}{t_s}\right)^{2(n-1)} \\ & \left. + 2\frac{n(n-1)}{t_s^2}\left(1 - \frac{t}{t_s}\right)^{n-2}\right]. \end{aligned} \tag{18}$$

One bears in mind the scale factor (3), the mass density (17), and the pressure (18).

4.1. Regularizing a Big-Bang singularity by varying G

If $G(t) \propto 1/(t^2)$, which is a faster decrease than in Dirac's Large Number Hypothesis (LNH) $G \propto 1/t$, and influences less the temperature of the Earth constraint (Teller 1948), then both divergence in ϱ and p are removed, though at the expense of having the "singularity" of strong gravitational coupling $G \to \infty$ at $t \to 0$. In the Dirac's case, only the singularity in ϱ can be removed.

4.2. Regularizing an SFS singularity by varying c

If

$$c(t) = c_0 \left(1 - \frac{t}{t_s}\right)^{\frac{p}{2}} , \tag{19}$$

then

$$p(t) = -\frac{c_0^2}{8\pi G}\left[\frac{m(3m-2)}{t^2}\left(1-\frac{t}{t_s}\right)^{p} - 6\frac{mn}{tt_s}\left(1-\frac{t}{t_s}\right)^{p+n-1} + 3\frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{p+2n-2} + 2\frac{n(n-1)}{t_s^2}\left(1-\frac{t}{t_s}\right)^{p+n-2}\right]. \tag{20}$$

and the singularity of pressure is regularized provided $p > 2 - n, (1 < n < 2)$.

Physical consequence of such a choice of regularization is that light eventually stops at singularity: $c(t_s) = 0$. Same happens in loop quantum cosmology (LQC), where we deal with the anti-newtonian limit $c = c_0\sqrt{1-\varrho/\varrho_c} \to 0$ for $\varrho \to \varrho_c$ with ϱ_c being the critical density (Cailleteau et al. 2012). The low-energy limit $\varrho \ll \varrho_0$ gives the standard case $c \to c_0 = \text{const}$. It also appears naturally in Magueijo (2001) model, in which black holes are not reachable since the light stops at the horizon (despite they possess Schwarzschild singularity). Both $c = 0$ and $c = \infty$ options are possible in Magueijo model.

4.3. No way of regularizing a w-singularity by varying c

In the limit $m \to 0$ of (3) we have an exotic singularity scale factor given by $a(t) = a_s \exp(1 - t/t_s)$, and from (17) and (18) we get

$$\varrho_{ex}(t) = \frac{3}{8\pi G(t)}\frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{2(n-1)} , \tag{21}$$

$$p_{ex}(t) = -\frac{c^2(t)}{8\pi G(t)}\left[3\frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{2(n-1)} + 2\frac{n(n-1)}{t_s^2}\left(1-\frac{t}{t_s}\right)^{n-2}\right] \tag{22}$$

so that

$$w_{ex}(t) = \frac{p_{ex}(t)}{\varepsilon_{ex}(t)} = -\left[1 + \frac{2}{3}\frac{n-1}{n}\frac{1}{\left(1-\frac{t}{t_s}\right)^n}\right] , \tag{23}$$

which is a w-singularity for $n > 2$ ($p = \varrho = 0$, $w_{ex} \to \infty$) (Dąbrowski & Denkiewicz 2009). Its regularization by varying $c(t)$ is impossible since there is no c-dependence here.

4.4. Regularizing an SFS singularity by varying G

If we assume that

$$G(t) = G_0\left(1-\frac{t}{t_s}\right)^{-r} , \tag{24}$$

($r = \text{const.}$, $G_0 = \text{const.}$) which changes (17) and (18) to

$$\varrho(t) = \frac{3}{8\pi G_0}\left[\frac{m^2}{t^2}\left(1-\frac{t}{t_s}\right)^{r} - \frac{2mn}{tt_s}\left(1-\frac{t}{t_s}\right)^{r+n-1} + \frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{r+2n-2}\right] , \tag{25}$$

$$p(t) = -\frac{c^2}{8\pi G_0}\left[\frac{m(3m-2)}{t^2}\left(1-\frac{t}{t_s}\right)^{r} - 6\frac{mn}{tt_s}\left(1-\frac{t}{t_s}\right)^{r+n-1} + 3\frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{r+2n-2} + 2\frac{n(n-1)}{t_s^2}\left(1-\frac{t}{t_s}\right)^{r+n-2}\right] . \tag{26}$$

From (25) and (26), it follows that an SFS singularity ($1 < n < 2$) is regularized by varying gravitational constant when $r > 2 - n$, and an FSF singularity ($0 < 1 < n$) is regularized when $r > 1 - n$. On the other hand, assuming that we have an SFS singularity and that $-1 < r < 0$, we get that varying G may change an SFS singularity onto a stronger FSF singularity when $0 < r + n < 1$.

5. Conclusions

Our proposal was to investigate how the standard and exotic FRW singularities are influenced by varying physical constants. In particular, we were looking for the answer if it was possible to "regularize" (remove infinities) or change these singularities and what were the physical consequences of such an action, because what we faced was often the new singularity in a physical constant/field which acted to remove/change the type of singularity.

We have shown that in order to regularize an SFS or an FSF singularity by varying $c(t)$, the light should slow and eventually stop propagating at a singularity. Similar effects were found in loop quantum cosmology (LQC) as well as in VSL theory for Schwarzschild horizon, where the speed of light was going either zero or to infinity at $r = r_s$. An observer could not reach this surface even in his finite proper time.

In order to regularize an SFS or an FSF by varying gravitational constant $G(t)$, the strength of gravity has to become infinite at singularity. It seems reasonable because of the requirement to overcome an infinite (anti-)tidal forces. On the other hand, it makes another singularity - a singularity of strong coupling for a physical field such as $G \propto 1/\Phi$. Such problems were already dealt with in superstring and brane cosmology where both the curvature singularity and a strong coupling singularity appeared (and requires special choice of coupling, or the application of quantum corrections).

Acknowledgements. This project was financed by the National Science Center Grant DEC-2012/06/A/ST2/00395.

References

Albrecht, A., & Magueijo, J. 1999, Phys. Rev. D, 59, 043516
Balcerzak, A., & Dąbrowski, M.P. 2006, Phys. Rev. D, 73, 101301(R)
Barrow, J.D. 2004, Class. Quantum Grav., 21, L79
Barrow, J.D. 2004, Class. Quantum Grav., 21, 5619
Bouhmadi-Lopez M., et al. 2008, Phys. Lett. B, 659, 1
Cailleteau, T., et al. 2012, Class. Quantum Grav., 29, 095010
Caldwell, R.R. 2002, Phys. Lett. B, 545, 23
Dąbrowski, M.P., & Denkiewicz, T. 2009, Phys. Rev. D, 79, 063521
Dąbrowski, M.P., & Denkiewicz, T. 2010, AIP Conf. Proc., 1241, 561
Dąbrowski, M.P., & Marosek, K. 2013, JCAP, 2, 12
Denkiewicz, T., et al. 2012, Phys. Rev. D, 85, 083527
Fernandez-Jambrina, L., & Lazkoz, R. 2006, Phys. Rev. D, 74, 064030
Frampton, P.H., et al. 2011, Phys. Rev. D, 84, 063003
Frampton, P.H., et al. 2012, Phys. Rev. D, 85, 083001
Gorini, et al. 2004, Phys. Rev. D, 69, 123512
Houndjo, M.J.S. 2010, Europhys. Lett., 92, 10004
Królak, A. 1988, Class. Quantum Grav., 3, 267
Loeb, A. 1998, ApJ , 499, L11
Magueijo, J. 2000, Phys. Rev. D, 62, 103521
Magueijo, J. 2001, Phys. Rev. D, 63, 043502
Nojiri, S., et al. 2005, Phys. Rev. D, 71, 063004
Perlmutter, S., et al. 1999, ApJ, 517, 565
Sandage, A. 1962, ApJ, 136, 319
Tegmark, M., et al. 2004, Phys. Rev. D, 69, 103501
Teller, E. 1948, Phys. Rev., 73, 801
Tipler, F. 1977, Phys. Lett. A, 64, 8
Vielzeuf, P., & Martins, C. 2012, Phys. Rev. D, 85, 087301
Webb, J., et al. 1999, Phys. Rev. Lett., 82, 884
Yurov, A.V. 2010, Phys. Lett. B, 689, 1