



Testing a non-minimal coupling between matter and curvature

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Abstract. One of the most interesting and current phenomenological extensions of General Relativity is the so-called $f(R)$ class of theories; a natural generalization of this includes an explicit non-minimal coupling between matter and curvature. The purpose of this work is to present a unified view of the applicability of the latter to various contexts, ranging from astrophysical matter distributions to a cosmological setting. Various results are discussed, including the impact of this non-minimal coupling in the choice of Lagrangian density, a mechanism to mimic galactic dark matter and a Cosmological Constant at an astrophysical scale, the possibility of accounting for the accelerated expansion of the Universe and modifications to post-inflationary reheating. The equivalence between a model exhibiting a non-minimal coupling and multi-scalar-theories is also discussed.

1. Introduction

Cosmology currently faces two outstanding challenges: the origin of dark energy and dark matter. Research on these dark components of the Universe may be roughly divided into two fields: one assumes that Einstein's General Relativity (GR) is valid at the relevant energy scales, thus focusing on the search for the missing matter or energy contribution; the other posits instead that dark matter and energy merely reflect deviations of gravity from GR.

When addressing dark matter, the first approach relies fundamentally on the characterization of additional matter species, arising mostly from extensions to the Standard Model of particles and fields, collectively dubbed as weak-interacting massive particles (WIMPs): these include *e.g.* neutralinos, which arise naturally from supersymmetry, or the axion, de-

rived from an elegant solution to the strong CP problem (Feng, 2010).

Similarly, the nature of dark energy usually involves the addition of a new scalar field, which slow-rolls down an adequate potential — quintessence (Copeland et al., 2006). A putative unification of dark matter and energy has also been suggested, by resorting to a scalar field (Bertolami & Rosenfeld, 2008) or an exotic equation of state (EOS), such as the generalized Chaplygin gas (Kamenshchik et al., 2001; Bilić et al., 2002; Bento et al., 2002).

The second approach follows a quite different route, by assuming that GR itself is at fault: instead of dark matter or energy, one may consider that dark gravity is at play. Thus, several attempts to generalize GR have surfaced: these are usually phenomenological in nature, positing putative low-energy modifications of Einstein's theory that should be derived from a yet unknown high-energy, fundamental theory of gravitation. In a cosmological context,

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modifications of the Friedmann equation to include higher order terms in the energy density ρ have been proposed (Maartens, 2004; Freese & Lewis, 2002; Carroll et al., 2004), as well as considerations of the impact of a van der Waals EOS for matter (Capozziello et al., 2002).

Another widely discussed candidate is the MODified Newtonian Dynamics (MOND), a non-relativistic modification of the Poisson equation, which becomes $\nabla[\mu(\nabla\Phi/a_0)\nabla\Phi] = 4\pi G\rho$, where Φ is the gravitational potential, ρ the density and $\mu(x)$ a function satisfying $\mu(x) \sim x$ for $x \ll 1$ and $\mu(x) \sim 1$ for $x \gg 1$. By introducing the typical acceleration $a_0 \sim 10^{-10} m/s^2$, MOND allows a simple derivation of the flattening of the rotation curves of galaxies, thus accounting for the missing “dark” matter challenge (Milgrom, 1983).

However, MOND has been shown to suffer from one outstanding issue, namely the inability to explain the gravitational profile of the so-called Bullet cluster (Clowe et al., 2006) — although a proposal resorting to heavy neutrinos with a mass close to ~ 2 eV has been put forward (Angus et al., 2006, 2007). On the theoretical side, the current relativistic formulation of MOND, dubbed Tensorial-Vector-Scalar (TeVeS) theory, resorts to two scalar (one dynamic and one auxiliary) and one vector field, together with a physical metric $\tilde{g}_{\mu\nu}$ used to couple with matter (Bekenstein, 2004): this formulation may scarcely be considered an elegant one, and one is tempted to call these extra fields as dark matter in another guise.

Changes to the action functional provide a simple pathway for implementing extensions to GR: a straightforward approach replaces the linear scalar curvature R term in the Einstein-Hilbert action with a function $f(R)$ (Capozziello et al., 2005; Allemandi et al., 2004) (see (Felice & Tsujikawa, 2010) for a review); more evolved dependences may also be explored, as considered in Gauss-bonnet models (Lovelock, 1971).

This class of $f(R)$ theories enjoys considerable success in several fronts, including the puzzle of the missing “dark” matter in galaxies and clusters (Capozziello et al., 2007, 2009a), as well as the nature of “dark” energy (Capozziello et al., 2003); the early pe-

riod of rapid expansion of the universe is described by the Starobinsky inflationary model $f(R) = R + \alpha R^2$ (Starobinsky, 1980); local Solar system impact and the related parameterized post-Newtonian (PPN) metric formalism have also been studied (Capozziello et al., 2009b).

The mathematical equivalence between $f(R)$ and scalar-tensor theories has proven very profitable, given the possibility of comparing results between the two types of model (Felice & Tsujikawa, 2010). Further insight is gained from considering the so-called Palatini formulation, where both the metric and the affine connection are taken as independent variables (Sotiriou et al., 2006) — an alternate to the usual metric affine connection approach, where the affine connection is taken *a priori* as depending on the metric.

Given the proficuous results arising from $f(R)$ theories, one is naturally tempted to further generalize this model — in the process extending its explanatory capability. Thus, another interesting possibility has been subject to scrutiny: not only that the curvature is non-trivial in the Einstein-Hilbert Lagrangian, but also that the coupling between matter and geometry is non-minimal (Bertolami et al., 2007), as written below:

$$S = \int [\kappa R + f_2(R)\mathcal{L}] \sqrt{-g}d^4x \quad (1)$$

where $\kappa = c^4/16\pi G$.

The purpose of this work is to present an unified view of several results that arise from this framework in a wide range of scenarios (Bertolami & Páramos, 2008, 2010, 2011; Bertolami et al., 2008, 2010, 2011) ; these were developed in collaboration with O. Bertolami, F. S. N. Lobo and P. Frazão.

2. The model

Variation with respect to the action Eq. (1) yields the modified Einstein field equations,

$$2(\kappa F_1 + F_2\mathcal{L})G_{\mu\nu} = 2\Delta_{\mu\nu}(\kappa F_1 + F_2\mathcal{L}) - [\kappa(F_1R - f_1) + F_2\mathcal{L}R]g_{\mu\nu} + f_2T_{\mu\nu} \quad (2)$$

where one defines $\Delta_{\mu\nu} \equiv \nabla_\mu\nabla_\nu - g_{\mu\nu}\square$ and $F_i(R) \equiv f'_i(R)$. As expected, GR is recovered

by setting $f_1(R) = R$ and $f_2(R) = 1$. By taking the trace of the above, one gets

$$\kappa(F_1 R - 2f_1) + F_2 \mathcal{L} R = \frac{1}{2} f_2 T - 3\Box(\kappa F_1 + F_2 \mathcal{L}) \quad (3)$$

where T is the trace of the energy-momentum tensor. The usual $f(R)$ theories are recovered by setting $f_2(R) = 1$, so that Eq. (3) yields an algebraic relation for $R = R(T)$. However, if one considers a non-minimal coupling $f_2(R) \neq 1$, the above becomes a differential equation: in particular, this enables the possibility that, given a rapidly varying $F_2 \mathcal{L}$, the above may lead to high curvatures even if T is low.

2.1. Non-conservation of energy

The Bianchi identities may be used to derive the non-(covariant) conservation of the energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = \frac{F_2}{f_2} (g^{\mu\nu} \mathcal{L} - T^{\mu\nu}) \nabla_\mu R \quad (4)$$

In the absence of a non-minimal coupling, $f_2(R) = 1$, one recovers the covariant conservation of the energy-momentum tensor.

Eq. (4) may be rewritten as an extra force imparted on test particles, so that its trajectory will deviate from geodesical motion. Thus, the Equivalence Principle may be broken if the *r.h.s.* of the last equation varies significantly for different matter distributions.

In Jordan-Brans-Dicke theories, the curvature appears coupled to a scalar field (Damour & Esposito-Farèse, 1992), leading to expressions similar to Eq. (4), due to the energy exchange between matter and the latter. However, this non-conservation may be transformed away by a suitable conformal transformation to the Einstein frame, where the curvature appears uncoupled (Faraoni et al., 1999). By the same token, one could assume that the non-conservation law Eq. (4) is only an artifact of the particular frame adopted, not a physically significant feature of the model.

However, performing the conformal transformation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = f_2 g_{\mu\nu}$, one finds that energy conservation is attained, $\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0$, only

if $\tilde{T}^{\mu\nu} = f_2^{-2} T^{\mu\nu}$ and $2\mathcal{L} = T$. Unless this relation holds for all matter species, energy conservation is indeed broken.

Thus, it suffices to use one example that breaks this relation: taking the case of a perfect fluid, with a Lagrangian density $\mathcal{L} = -\rho$ (Bertolami et al., 2008), then one obtains $p = -\rho/3$ — signaling a perfect fluid with negative pressure (but different from a cosmological constant, where one has $p_\Lambda = -\rho_\Lambda$); if one instead opts for the Lagrangian density $\mathcal{L} = p$ (Sotiriou & Faraoni, 2008), then $p = \rho$ is obtained — the EOS for ultra-stiff matter.

The discussion of which form for \mathcal{L} is correct in the presence of a non-minimal coupling is addressed in the following section; nevertheless, it is clear that one cannot recover energy conservation for a perfect fluid — and, by extension, for all matter species: this non-conservation law is indeed a fundamental property of the model under scrutiny.

3. Equivalence with multi-scalar-tensor models

Through a suitable conformal transformation, the usual $f(R)$ theories can be rewritten as GR with an added scalar field contribution, which is dynamically identified with the curvature, $\phi = R$. Similarly, the discussed non-minimally coupled model Eq. (1) can be recast as a multi-scalar field theory, with two scalar fields, albeit a “physical” metric remains in the matter Lagrangian (Bertolami & Páramos, 2008). Indeed, by performing a conformal transformation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \exp[(2/\sqrt{3})\varphi^1] g_{\mu\nu}$, the equivalent action is obtained,

$$S = \int \sqrt{-\tilde{g}} d^4x \left[f_2(\varphi^2) e^{-\frac{2}{\sqrt{3}}\varphi^1} \mathcal{L}(g_{\mu\nu}, \chi) + 2\kappa \left(R - 2\tilde{g}^{\mu\nu} \sigma_{ij} \varphi_{,\mu}^i \varphi_{,\nu}^j - 4U \right) \right] \quad (5)$$

where χ denotes all matter fields, φ^1 and φ^2 are scalar fields related to the scalar curvature and matter Lagrangian density through

$$\varphi^1 = \frac{\sqrt{3}}{2} \log \left[\frac{F_1(R) + F_2(R)\mathcal{L}}{2\kappa} \right] \quad (6)$$

and $\varphi^2 = R$; σ_{ij} is the field-metric

$$\sigma_{ij} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad (7)$$

and the potential is given by

$$U(\varphi^1, \varphi^2) = \frac{1}{4} \exp\left(-\frac{2\sqrt{3}}{3}\varphi^1\right) \times \left[\varphi^2 - \frac{f_1(\varphi^2)}{2\kappa} \exp\left(-\frac{2\sqrt{3}}{3}\varphi^1\right)\right], \quad (8)$$

Notice that any use of the metric in this Lagrangian density (contractions in kinetic terms, *etc.*) is done not with $\tilde{g}_{\mu\nu}$, but with the physical metric $g_{\mu\nu} = \exp[-(2/\sqrt{3})\varphi^1]\tilde{g}_{\mu\nu}$.

4. Lagrangian density of a perfect fluid

The argument of the preceding section serves to show another striking feature of non-minimally coupled models: the Lagrangian density of matter appears explicitly in the field equations. In particular, one focuses on the case of a perfect fluid, given its ubiquity as a useful description for standard matter. Its energy-momentum tensor is the familiar form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (9)$$

where ρ is the energy density, p the pressure and u_μ the four-velocity (with $u_\mu u^\mu = -1$); its trace is $T = 3p - \rho$. The associated Lagrangian density is less discussed, precisely because it is absent from the field equations of GR: in applications of GR, the choice of Lagrangian density is mostly irrelevant, while it is of the utmost importance when considering a non-minimal coupling (Bertolami et al., 2008).

The identification $\mathcal{L} = p$ was first advanced in (Seliger & Whitham, 1968), with a relativistic generalization in (Schutz, 1970). Much later, (Brown, 1993) showed that this choice is equivalent to $\mathcal{L} = -\rho$, complemented by a suitable set of thermodynamical potentials and Lagrangian multipliers $\phi^\mu = \varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha_{,\mu}^A$; these enable the relativistic thermodynamical relations via a current term $J_\mu \phi^\mu$ in the action, where J_μ is the vector density, *i.e.* the

flux vector of the particle number density. In (Hawking & Ellis, 1973), an isentropic perfect fluid is described via $\mathcal{L} = -\rho$.

The equivalence between occurs on-shell, by substituting the field equations derived from the matter action

$$S_m = \int (-\sqrt{-g}\rho + J_\mu \phi^\mu) d^4x, \quad (10)$$

back into the action functional and reads the resulting on-shell Lagrangian density $\mathcal{L}_1 = p$. Similarly, the action may be rewritten so that the on-shell Lagrangian density reads $\mathcal{L}_2 = na$, where $n = |J|/\sqrt{-g}$ is the particle number density and $a(n, T) = \rho(n)/n - sT$, where s is the entropy per particle and T is the temperature. Lastly, one may remove the current term $J_\mu \phi^\mu/\sqrt{-g}$ from Eq. (10), thus obtaining the on-shell Lagrangian density $\mathcal{L}_3 = -\rho$ through the addition of adequate surface terms.

Indeed, (Seliger & Whitham, 1968) did not argue that $\mathcal{L}_1 = p$ was the bare (as opposed to on-shell) Lagrangian density for a perfect fluid, but resorted to a much more evolved action functional: this simple identification with the pressure also arose only after the resulting field equations were substituted into it.

With the above in mind, one now ascertains how should this procedure be generalized to a non-minimally coupled scenario. This should affect the terms in Eq. (10) that are minimally coupled, so that the matter action becomes

$$S'_m = \int (-\sqrt{-g}f_2(R)\rho + J^\mu \phi_\mu) d^4x, \quad (11)$$

while the current term remains uncoupled (aside from the use of the metric to contract indexes). Varying the action with respect to the potentials included in ϕ_μ , one obtains

$$-f_2\mu U_\mu = \phi_\mu, \quad (12)$$

$$T = \frac{1}{n} \frac{\partial \rho}{\partial s} \Big|_n = \frac{1}{f_2(R)} \theta_{,\mu} U^\mu,$$

where μ is the chemical potential and θ is a scalar field (included in ϕ_μ) whose equation of motion imposes the entropy exchange constraint $(sJ^\mu)_{,\mu} = 0$. Thus, the non-minimal coupling of curvature to matter is reflected in both the velocity and the temperature identification.

By substituting the modified equations of motion into action (11), an on-shell Lagrangian density $\mathcal{L}_1 = p$ may be read, as in GR. The addition of surface integrals also yields the discussed $\mathcal{L}_2 = -\rho$ and $\mathcal{L}_3 = -na$.

Albeit the action may adopt distinct on-shell forms, this does not translate into a direct equivalence between them: only the original bare Lagrangian density \mathcal{L}_0 should be inserted into the field equations Eq. (2). However, this bare \mathcal{L}_0 should not appear into the non-conservation law: indeed, when deriving Eq. (4), the current term (which is not coupled with the metric) is dropped and one is indeed left with $\mathcal{L}_2 = -\rho$.

Notwithstanding the above, one can adopt a simpler stance regarding the choice of the Lagrangian density $\mathcal{L} = -\rho$ instead of $\mathcal{L} = p$: if a dust distribution is to be considered — that is, a perfect fluid with negligible pressure and corresponding energy-momentum tensor $T_{\mu\nu} = \rho g_{\mu\nu}$ —, it appears unnatural to take a vanishing quantity as Lagrangian density.

Ending this section, one remarks that if a matter form is actually described by two independent bare Lagrangian densities leading to different dynamical behaviour of Eq. (2) and (4), then only confrontation with observation can ascertain which is the correct description.

5. Dark matter mimicking

One now reviews a mechanism that mimics the missing dark matter revealed through the flattening of galaxy rotation curves (Bertolami & Páramos, 2010). In order to isolate the effect of the non-minimal coupling, one sets $f_1(R) = 1$ and assumes a power-law

$$f_2(R) = 1 + \left(\frac{R}{R_n}\right)^n, \quad (13)$$

Since the dark matter contribution is dominant at large distances *i.e.* low curvatures, a negative exponent n is expected.

Inserting this into the modified field Eq. (2), together with the Lagrangian density $\mathcal{L} = -\rho$ for a dust distribution with $p = 0$, yields

$$\left[1 - n \left(\frac{R}{R_n}\right)^n \frac{\rho}{\kappa R}\right] R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \quad (14)$$

$$\left[1 + \left(\frac{R}{R_n}\right)^n\right] \frac{\rho}{2\kappa} U_\mu U_\nu - n \Delta_{\mu\nu} \left[\left(\frac{R}{R_n}\right)^n \frac{\rho}{\kappa R}\right].$$

At large distances, normal matter is subdominant and the trace Eq. (3) reads

$$R = (1 - 2n) \left(\frac{R}{R_n}\right)^n \frac{\rho}{2\kappa} - 3n \square \left[\left(\frac{R}{R_n}\right)^n \frac{\rho}{\kappa R}\right]. \quad (15)$$

Inspection shows that an exact solution is obtained if the last term vanishes,

$$R = R_n \left[(1 - 2n) \frac{\rho}{\rho_n}\right]^{1/(1-n)}. \quad (16)$$

defining the characteristic density $\rho_n = 2\kappa R_n$.

A more evolved study of Eq. (15) shows that the most general solution oscillates around the one above: this makes the gradient term actually dominate Eq. (15), and also allows for a perturbative $f_2(R) \ll 1$. As numerical results show, one may disregard these oscillations and simply consider Eq. (16) (Bertolami & Páramos, 2010). A perturbative non-minimal coupling is paramount: it makes the mimicking mechanism satisfy both that the weak, strong, null and dominant energy conditions; grants immunity against Dolgov-Kawasaki instabilities (Dolgov & Kawasaki, 2003); and makes the extra force arising from Eq. (4) very small.

Instead of solving Eq. (14), obtaining a modified gravitational potential and then reading the mimicked dark matter contribution, one interprets the additional curvature obtained from the non-minimal coupling as due to the latter's density profile,

$$\rho_{dm} \equiv 2\kappa R = \rho_n \left[(1 - 2n) \frac{\rho}{\rho_n}\right]^{1/(1-n)}. \quad (17)$$

Thus, one obtains a direct translation between a visible matter profile ρ and the mimicked dark matter distribution ρ_{dm} , with n related to the large radius behaviour of both contributions.

The mimicked dark matter may be further characterized by inserting Eq. (16) into Eq. (14), and reading the obtained terms as due to a corresponding energy-momentum tensor: this allows one to obtain the EOS parameter for dark matter, $\omega = p_{dm}/\rho_{dm} = n/(1-n)$: for a negative exponent n , one has $\omega < 0$, hinting at a possible cosmological role in enabling an

accelerated expansion of the Universe, as will be discussed in the following section.

In order to fit the galaxy rotation curves of several galaxies (NGC 2434, 5846, 6703, 7145, 7192, 7507 and 7626, which are almost spherical type E0 galaxies with well determined rotation curves (Gerhard et al., 2000), one resorts to the Hernquist profile for visible matter (Hernquist, 1990), which behaves as

$$\rho(r) \sim \frac{a}{r} \left(1 + \frac{r}{a}\right)^{-3}, \quad (18)$$

and the Navarro-Frenk-White (Navarro et al., 1995) and isothermal sphere profiles for dark matter: the latter enables a perfectly flat rotation curve, while the former is favored by numerical simulations:

$$\rho_{IS}(r) \sim r^{-2}, \quad \rho_{NFW}(r) \sim \frac{a}{r} \left(1 + \frac{r}{a}\right)^{-2}. \quad (19)$$

with a signalling the transition between inner and outer slope profiles. Given the outer slope behaviours $\rho(r) \sim r^{-4}$, $\rho_{IS}(r) \sim r^{-2}$ and $\rho_{NFW}(r) \sim r^{-3}$, the relation $\rho_{dm}(r) \propto \rho(r)^{1/(1-n)}$ yields the exponents $n_{IS} = -1$ and $n_{NFW} = -1/3$ that translate the Hernquist profile into the two above. Thus, one is led to consider the composite non-minimal coupling

$$f_2(R) = 1 + \frac{R_{-1}}{R} + \sqrt[3]{\frac{R_{-1/3}}{R}}. \quad (20)$$

The values of the model parameters R_{-1} and $R_{-1/3}$ are obtained from fitting the result of the numerical integration of Eq. (14) to the available galaxy rotation curves: by doing so, one concludes that $R_{-1} \sim 1/(16.8 \text{ Gpc})^2$ and $R_{-1/3} \sim 1/(1.45 \times 10^6 \text{ Gpc})^2$. Fig. 1 shows the fit to NGC 5846, one of the galaxies addressed in (Bertolami & Páramos, 2010).

Allowing for individual fits of these parameters to each of the considered galaxies shows that there is some dispersion around these orders of magnitude, which could be due to deviations from sphericity, poor choice of visible or dark matter density profiles, unaccounted effect of a non-trivial $f_1(R) \neq R$ term or the non-minimal coupling with the electromagnetic sector (thus enabling a dependence on luminosity) *etc.*. Nevertheless, the quality of

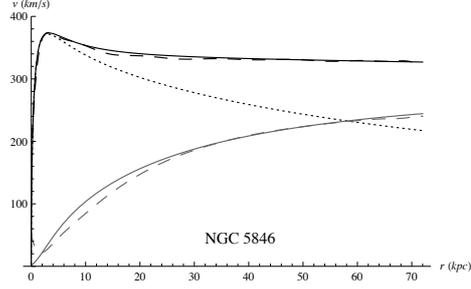


Fig. 1. Observed rotation curve (dashed full), decomposed into visible (dotted) and dark matter (dashed grey) contributions (Gerhard et al., 2000), with mimicked dark matter profile (full grey) and resulting full rotation curve (full).

the obtained fits and the elegance of the mimicking mechanism support its ability to account for the missing dark matter puzzle.

6. Accelerated expansion of the Universe

This section discusses the possible use of a non-minimal coupling to describe the current phase of accelerated expansion of the Universe (Bertolami et al., 2010). One focuses on a constant deceleration parameter $q \equiv -\ddot{a}/(\dot{a})^2 = 1/\beta - 1$, which translates into a power-law expansion with the scale factor evolving as $a(t) = a_0(t/t_0)^\beta$, with $t_0 = 13.73 \text{ Gy}$.

This hints at the use of a power-law for the non-minimal coupling also, as considered in the previous section. Thus, one takes $f_2(R) = 1 + (R/R_n)^n$, with the exponent n assumed to be negative — so that the accelerated expansion phase appears at late times, when $R \ll R_n$.

This accelerated expansion is obtained in a rather straightforward fashion: assuming a flat Friedmann-Robertson-Walker metric with line element $ds^2 = -dt^2 + a^2 dV^2$ and that matter is described by a perfect fluid with density ρ and pressure p , one has $T_{00} = \rho$ and $T_{rr} = a^2 p$. One then uses $\mathcal{L} = -\rho$ and Eq. (4) to ascertain that energy is covariantly conserved in a cosmological context,

$$\nabla_\mu T^{\mu 0} = \frac{F_2}{f_2} (g^{00} \mathcal{L} - T^{00}) \dot{R} = 0 \rightarrow \quad (21)$$

$$\dot{\rho} + 3H\rho = 0 \rightarrow \rho(t) = \rho_0 \left(\frac{t_0}{t}\right)^{3\beta} .$$

where $\rho_0 = \Omega_m \rho_{crit}$, with $\Omega_m \sim 0.3$ the relative matter density and $\rho_{crit} \sim 10^{-26} \text{ kg/m}^3$ the critical density (Spergel et al., 2007).

One uses this result together with Eq. (2) to compute the modified Friedmann equation

$$H^2 = \frac{1}{6\kappa}(\rho + p + \rho_c + p_c) , \quad (22)$$

where the additional density ρ_c and pressure p_c terms are introduced,

$$\rho_c = -6\rho_0\beta \frac{1 - 2\beta + n(5\beta + 2n - 3)}{\left(\frac{t}{t_0}\right)^{3\beta} \left(\frac{t}{t_2}\right)^{2n} [6\beta(2\beta - 1)]^{1-n}} , \quad (23)$$

$$p_c = -2\rho_0 n \frac{2 + 4n^2 - \beta(2 + 3\beta) + n(8\beta - 6)}{\left(\frac{t}{t_0}\right)^{3\beta} \left(\frac{t}{t_2}\right)^{2n} [6\beta(2\beta - 1)]^{1-n}} .$$

defining $t_n \equiv R_n^{-1/2}$ and taking the weak regime $F_2\rho \ll \kappa$ (Bertolami et al., 2010).

Given the Hubble parameter $H(t) \equiv \dot{a}/a = \beta/t$, the *l.h.s.* of the Friedmann Eq. (22) falls as t^{-2} , so that comparing with the above gives a relation between the exponents β and n :

$$3\beta + 2n = 2 \rightarrow \beta = \frac{2}{3}(1 - n) . \quad (24)$$

Thus, one concludes that any negative exponent n will yield an accelerating Universe with $\beta, q > 0$. One may also compute the deceleration parameter and equation of state parameter for the non-minimally coupled contribution,

$$q = -1 + \frac{3}{2(1-n)} , \quad \omega = \frac{n}{1-n} . \quad (25)$$

The latter is the same form found in the previous section. By fitting the numerical solution of Eq. (22) to the evolution profile of $q(z)$ (Gong & Wang, 2007), one obtains the best fit values $n = -10$ and $t_{-10} = t_0/2$, depicted in Fig. 2.

This value for the exponent n is very distinct from the $n_{IS} = -1$ and $n_{NFW} = -1/3$ scenarios considered in the previous section. These have no cosmological impact on the current scenario, given the broad difference between the relevant timescale t_0 and those obtained from $t_{-1} \equiv R_{-1}^{-1/2}$ and $t_{-1/3} \equiv R_{-1/3}^{-1/2}$. Its

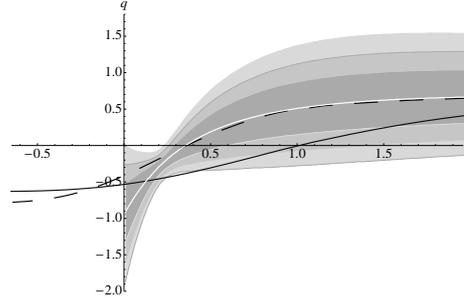


Fig. 2. Evolution of the deceleration parameter $q(z)$ for $n = -4$, $t_2 = t_0/4$ (full) and $n = -10$, $t_2 = t_0/2$ (dashed); 1σ , 2σ and 3σ allowed regions are shaded, white line gives best fit (Gong & Wang, 2007).

value $n = -10$ may also be regarded as somewhat unnatural, but is highly dependent on the $q(z)$ profile: Fig. 2 shows that $n = -4$, $t_2 = t_0/4$ also yields a curve within the allowed 3σ region. Despite this caveat, one concludes that a non-minimal coupling may be used to describe the accelerated expansion of the Universe.

7. Matter solution with constant curvature

The main result of the previous section is embodied by Eq. (25), which directly translates the exponent n of a power-law coupling between curvature and matter to the deceleration parameter q . However, it is trivial to check that one cannot produce a de Sitter Universe, characterized by $q = -1$, as this would require $n \rightarrow \infty$ — so that a Cosmological Constant (CC) Λ with EOS $p_\Lambda = -\rho_\Lambda$ cannot be recovered. One could argue that this hints at the need to consider a different form for $f_2(R)$ — perhaps an exponential, given the behaviour of the scale factor $a(t) = a_0 \exp(H_0 t)$.

However, as was shown in (Bertolami et al., 2010), it turns out that a de Sitter phase is impossible to obtain in a cosmological context, fundamentally because the curvature is given by the CC, $R = 4\Lambda$, and is thus constant: since the density still evolves as $\rho(t) \propto a^{-3} \sim \exp(-3H_0 t)$, the terms arising from the non-minimal coupling in the modified Friedmann cannot be interpreted as an additional density

and pressure due to a CC, $p_{CC} = -\rho_{CC}$. One may simplistically state that, since the $f_2(R)\mathcal{L}$ term in Eq. (1) cannot be made constant, a CC is unattainable.

With the above in mind, this section briefly discusses the possibility that a CC is not directly generated in a cosmological context, but arises due to an averaging of several constant curvature “bubbles” (Bertolami & Páramos, 2011). This does not aim at solving the Cosmological Constant problem: the $\sim 10^{120}$ discrepancy between its expected value and that needed to account for the current accelerated expansion, but draws a mechanism to produce the latter (for a review, see Bertolami, 2009).

Notice that a constant curvature $R = R_0$ implies that all terms involving f_2 and its derivatives become constants; thus, the actual form of f_2 is shown to be only constrained by $F_2(4\Lambda)\Lambda \ll f_2(4\Lambda)$. In particular, the linear or power-law non-minimal couplings considered in this work are compatible with this relation.

One uses a static Birkhoff metric with line element $ds^2 = -e^{2\nu(r)}dt^2 + e^{2\sigma(r)}dr^2 + d\Omega^2$; matter is assumed to be described by a perfect fluid with constant EOS parameter $\omega = p/\rho$. The Einstein field Eq. (2) with $f_1(R) = R$ thus leads to the differential equation,

$$\begin{aligned} 0 = \rho'' - \frac{2}{r}\rho' + \frac{1+\omega}{2(KR_0r)^2} \frac{f_2}{F_2} \rho(\rho')^2 - & (26) \\ - \frac{1}{3r^3} \left(\frac{1+\omega}{2(KR_0)^2} \right)^2 \left(\frac{f_2}{F_2} \right)^2 \rho^2(\rho')^3 + \\ + \frac{1}{6(KR_0)^2 r} (\rho')^3 \left(\frac{2}{r^2} - R_0 + \frac{1+\omega}{2} \frac{f_2}{F_2} \right), \end{aligned}$$

where K is an integration constant. Imposing continuity with a Schwarzschild-de Sitter metric at the boundary of the matter distribution $r = r_s$, one obtains

$$K = \frac{\rho'(r_s)}{R_0} \sqrt{\frac{1}{3r_s^2} - \Lambda}. \quad (27)$$

The boundary radius r_s is related to the total mass of the matter distribution via

$$r_s \sim \sqrt{\frac{\rho_H}{\rho_c}} 11.4 \text{ Mpc}, \quad (28)$$

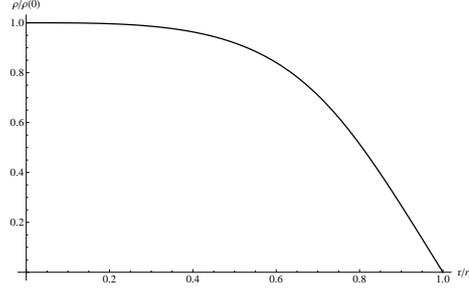


Fig. 3. Rescaled density profile arising from $R_0 = 4\Lambda$.

where $\rho_H = 1 \text{ H atom/cm}^3 = 1.660 \times 10^{-21} \text{ kg/m}^3$. The (rescaled) numerical solution to Eq. (26) when $R_0 = 4\Lambda$ (*i.e.* the local curvature yields a CC) is shown in Fig. 3. The typical distance between galaxies is of the order of $r_s \sim 1 \text{ Mpc}$ (*e.g.* Andromeda is 778 Kpc away), so that $\rho_c \approx 130 \text{ H atom/cm}^3$ — close to the lower bound of *HII* star-forming regions. The interstellar medium attains densities up to 10^4 H atom/cm^3 , allowing for a radius of the constant curvature bubble down to $\sim 100 \text{ Kpc}$. Thus, one concludes that it is possible to obtain a smooth constant curvature matter solution $R_0 = 4\Lambda$ compatible with the expected values for realistic density profiles.

7.1. Averaging mechanism

Having obtained a feasible description of one constant curvature “bubble”, one now briefly discusses how several of these may average and produce a CC at cosmological scales. One first argues that a universal value of $R_0 = 4\Lambda$ is to be expected in all local “bubbles”, as it should be derived from the non-minimal coupling $f_2(R)$ itself: a full account of the dynamics of the model, obtained by allowing the curvature to vary, should lead to a relaxation towards a constant value R_0 arising from the modified field Eq. (2), if the composite effect of a non-trivial curvature term $f_1(R)$ and coupling $f_2(R)$ exhibits a suitable minimum, thus acting as an attractive potential.

This minimum $R_0(\rho)$ would naturally depend on the underlying density, suggesting a similarity with the so-called chameleon model

(Khoury & Weltman, 2004) — indeed the full Lagrangian $f_1(R) - f_2(R)\rho$ may be suggestively compared with the usual chameleon effective potential $V_{eff}(\phi) = V(\phi) + \rho \exp(\phi/M)$. At early times, local density and curvature solutions could be widely different from each other and from the constant curvature and specific matter distribution assumed here: however, extensive neighbour to neighbour interaction would favour similar density profiles, yielding a universal minimum R_0 . This would lead to a progressive decrease of the deceleration parameter towards $q = -1$, signaling the onset of a CC (Gong & Wang, 2007).

8. Post-inflationary reheating

In this section one discusses the possibility of using a non-minimal coupling to drive the reheating of the Universe after the inflationary stage occurring in its early times (Bertolami et al., 2011). One does not attempt to drive the dynamics of inflation through the effect of $f_2(R)$, but instead assumes the well-known Starobinsky inflation model (Starobinsky, 1980)— which resorts to a quadratic curvature term

$$f_1(R) = R + \frac{R^2}{6M^2}, \quad (29)$$

with $M \sim 3 \times 10^{-6} M_p$.

A non-minimal coupling is fundamental in the so-called preheating mechanism: the reheating of the ultracold post-inflationary Universe due to the explosive production of particles, occurring when the dynamics of a quantum scalar field χ endowed with a variable mass term of the form $m_{eff}^2 = m^2 + \xi R$ experience parametric resonance (Tsujikawa et al., 1999). More evolved couplings also lead to preheating, as found in Refs. (Charters et al., 2008; Lachapelle & Brandenberger, 2009).

This hints that the non-minimally coupled action (1) may generalize the preheating scenario. Given the form of the variable mass term m_{eff} , one assumes a linear coupling

$$f_2(R) = 1 + 2\xi \frac{R}{M^2}. \quad (30)$$

Since the curvature is coupled to matter and radiation, besides the scalar field χ , one must ensure that the cosmological dynamics are driven by the effect of the quadratic curvature term Eq. (29) alone — *i.e.* that the non-minimal coupling only intervenes during preheating, not before. As shown in (Bertolami et al., 2011), this implies a perturbative regime $f_2(R) \sim 1$ and that $1 < \xi < 10^4$, compatible with the weak bound $\xi \ll 10^{78}$ obtained from considerations on solar hydrostatic equilibrium (Bertolami e Páramos, 2008).

Decomposing the scalar field χ into its Fourier modes χ_k , one finds that they follow the differential equation

$$X_k'' + \left[\left(\frac{2k}{aM} \right)^2 + \left(\frac{2m}{M} \right)^2 - 3 \frac{H'}{M} - 9 \frac{H^2}{M^2} + \frac{\xi}{M^2} \left(\xi \frac{R'^2}{M^2} - 6 \frac{HR'}{M} - R'' \right) \right] X_k = 0. \quad (31)$$

using the redefinition $X_k \equiv a^{3/2} f_2^{1/2} \chi_k \sim a^{3/2} \chi_k$ and the new variable $2z = M(t - t_0) \pm \pi/2$ (depending on the sign of ξ).

After slow-roll, the Hubble parameter $H(t)$ experiences an oscillatory phase, so that $R \sim (2M^2/z) \cos(2z)$: the R'' term dominates and one rewrites the above as a Mathieu equation,

$$X_k'' + [A_k - 2q \cos(2z)] X_k = 0, \quad (32)$$

with

$$A_k = \left(\frac{2k}{aM} \right)^2 + \left(\frac{2m}{M} \right)^2, \quad q = \frac{4\xi}{z}. \quad (33)$$

Eq. (32) is the exact same form found in usual preheating (Tsujikawa et al., 1999): the quantum field χ experiences parametric resonance as the scale factor $a(t)$ increases, with massless particles produced for a coupling parameter as low as $\xi \gtrsim 3$, while massive particles require $\xi \gtrsim 10$ — well within the range $1 < \xi < 10^4$ discussed above. Thus, one finds that, as expected, a universal non-minimal coupling may successfully drive the reheating of the post-inflationary Universe.

9. Conclusions and outlook

A non-minimal coupling between geometry and matter covers a broad spectrum of applications in an elegant, natural way. Hopefully,

this hints at its relevance in the formulation of a fundamental quantum theory of gravity: future advances on the theoretical front should help clarify its nature.

The distinct expressions for the non-minimal coupling here considered may be regarded as an approximation to a more evolved forms for $f_2(R)$, each valid in a particular regime: early *vs.* late time, central *vs.* long range, it etc.. The latter could perhaps be written as a Laurent series,

$$f_2(R) = \sum_{n=-\infty}^{\infty} \left(\frac{R}{R_n} \right)^n, \quad (34)$$

so that one could chart yet unprobed terms of the above series by assessing the dynamics of other phenomena and environments, where distinct curvatures and densities are at play.

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