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# Time transfer function in static, spherically symmetric space-times

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**Abstract.** We outline of new method enabling to determine the time transfer function and the propagation direction of light rays in parametrized static, spherically symmetric space-times. Explicit results up and including the third order in the Schwarzschild radius are given.

## 1. Introduction

We present a new method for determining the time transfer function and the propagation direction of light rays in static, spherically symmetric space-times at any given order in the powers of $GM/c^2r$, $M$ being the mass of the body generating the gravitational field and $G$ the Newtonian gravitational constant. In contrast with the procedures developed in Le Poncin-Lafitte et al. (2004) and Teyssandier & Le Poncin-Lafitte (2008), this method is based on the null geodesic equations. The third-order terms are explicitly written. This study is motivated by the fact that a knowledge of the corrections of higher orders is indispensable for an in-depth discussion of the most accurate tests of the metric theories of gravity (see, e.g., Klioner & Zschocke 2010; Ashby & Bertotti 2010). As far as we know, the main result of this paper is new since the travel time of the photons was not determined in the previous works devoted to the third-order approximation (Sarmiento 1982; Keeton & Petters 2005).

## 2. Time transfer function in static, spherically symmetric space-times

Assuming that the gravitational field is generated by a static, spherically symmetric body, we consider a photon emitted at time $t_A$ from an observer at point $\boldsymbol{x}_A$ and received at time $t_B$ by an observer at point $\boldsymbol{x}_B$. The travel time $t_B - t_A$ of this photon is a function of $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, so we can put

$$t_B - t_A = \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B), \tag{1}$$

$\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ being called the time transfer function.

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In this paper, we restrict our attention to the determination of $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ because it is shown in Le Poncin-Lafitte et al. (2004) that the light propagation directions at points $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ are characterized by the triples given by the relations

$$\left(\frac{l_i}{l_0}\right)_A = c\frac{\partial \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)}{\partial x_A^i}, \qquad \left(\frac{l_i}{l_0}\right)_B = -c\frac{\partial \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)}{\partial x_B^i}, \tag{2}$$

where $l_0$ and $l_i$ are the covariant components of a vector tangent to the ray, that is a system of quantities defined by $l_\alpha = g_{\alpha\beta} dx^\beta / d\lambda$, $g_{\alpha\beta}$ denoting the components of the metric tensor and $\lambda$ an arbitrary parameter along the ray.

In what follows, the metric is written in isotropic coordinates:

$$ds^2 = \mathcal{A}(r)c^2dt^2 - \mathcal{B}^{-1}(r)\,\delta_{ij}dx^i dx^j. \tag{3}$$

Putting $m = GM/c^2$, we suppose that the potentials are given by expansions in powers of $m/r$:

$$\begin{aligned}\mathcal{A}(r) &= 1 - \tfrac{2m}{r} + 2\beta\tfrac{m^2}{r^2} - \tfrac{3}{2}\beta_3\tfrac{m^3}{r^3} + \beta_4\tfrac{m^4}{r^4} + \textstyle\sum_{n=5}^{\infty}\tfrac{(-1)^n n}{2^{n-2}}\beta_n\tfrac{m^n}{r^n}, \mathcal{B}(r)^{-1} \\ &= 1 + 2\gamma\tfrac{m}{r} + \tfrac{3}{2}\epsilon\tfrac{m^2}{r^2} + \tfrac{1}{2}\gamma_3\tfrac{m^3}{r^3} + \tfrac{1}{16}\gamma_4\tfrac{m^4}{r^4} + \textstyle\sum_{n=5}^{\infty}(\gamma_n - 1)\tfrac{m^n}{r^n},\end{aligned} \tag{4}$$

where $\beta, \beta_3, ..., \beta_n, ..., \gamma, \epsilon, \gamma_3, ..., \gamma_n, ...$ are generalized post-Newtonian parameters defined so as to have $\beta = \gamma = \epsilon = \beta_n = \gamma_n = 1$ in general relativity.

Assuming that $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ admits an expansion as follows

$$\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{|\boldsymbol{x}_B - \boldsymbol{x}_A|}{c} + \sum_{n=1}^{\infty} \mathcal{T}^{(n)}(\boldsymbol{x}_A, \boldsymbol{x}_B), \tag{5}$$

where $\mathcal{T}^{(n)}$ stands for the term of order $n$ in $G$, it is shown in Teyssandier & Le Poncin-Lafitte (2008) that each term $\mathcal{T}^{(n)}$ is given by an integral taken along the straight segment joining $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$. For $n = 1$, one recovers the well-known Shapiro term, namely

$$\mathcal{T}^{(1)}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{(\gamma + 1)m}{c} \ln\left(\frac{r_A + r_B + |\boldsymbol{x}_B - \boldsymbol{x}_A|}{r_A + r_B - |\boldsymbol{x}_B - \boldsymbol{x}_A|}\right), \tag{6}$$

and for $n = 2$, one obtains the simple expression

$$\mathcal{T}^{(2)}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{m^2}{r_A r_B}\frac{|\boldsymbol{x}_B - \boldsymbol{x}_A|}{c} \times \left[\frac{\kappa \arccos(\boldsymbol{n}_A . \boldsymbol{n}_B)}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|} - \frac{(\gamma + 1)^2}{1 + \boldsymbol{n}_A . \boldsymbol{n}_B}\right], \tag{7}$$

where

$$\boldsymbol{n}_A = \frac{\boldsymbol{x}_A}{r_A}, \quad \boldsymbol{n}_B = \frac{\boldsymbol{x}_B}{r_B}, \quad \kappa = \frac{8 - 4\beta + 8\gamma + 3\epsilon}{4}. \tag{8}$$

Nevertheless, determining the integrals yielding the quantities $\mathcal{T}^{(n)}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ requires more and more complex calculations as the order $n$ is increasing. So it is of interest to explore some alternative procedures, like the one which is outlined below.

## 3. Method of constrained integration

Using spherical coordinates $(r, \vartheta, \varphi)$ and choosing the axes in such a way that $\vartheta = \pi/2$ along the ray, the null geodesic equations may be written as

$$\frac{dt}{dr} = \pm \frac{1}{c\sqrt{\mathcal{A}(r)\mathcal{B}(r)}} \frac{r}{\sqrt{r^2 - b^2\mathcal{A}(r)\mathcal{B}(r)}} \tag{9}$$

and

$$\frac{d\varphi}{dr} = \pm' \frac{b}{r} \frac{\sqrt{\mathcal{A}(r)\mathcal{B}(r)}}{\sqrt{r^2 - b^2\mathcal{A}(r)\mathcal{B}(r)}}, \tag{10}$$

where $b$ is the impact parameter of the light ray (Chandrasekhar 1983), which may be considered as a constant of the motion (Teyssandier 2010).

In what follows, we may assume that the light ray does not pass through a periastron, since the well-known analytic extension theorem ensures that each formula giving $\mathcal{T}^{(n)}$ as a function of $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ is valid provided that the ray remains confined to a region of 'weak field' ($r \gg m$ at each point of the ray). As a consequence the signs in Eqs. (9) and (10) may be taken as positive without loss of generality, which implies $r_B > r_A$ and $\varphi_B > \varphi_A$. Then Eq. (9) yields

$$\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{c} \int_{r_A}^{r_B} \frac{r dr}{\sqrt{\mathcal{A}(r)\mathcal{B}(r)[r^2 - b^2\mathcal{A}(r)\mathcal{B}(r)]}}. \tag{11}$$

The expression of the impact parameter of the ray as a function of $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ may be obtained by solving for $b$ the 'constraint equation' obtained by integrating Eq. (10) along the light ray. On our assumptions, this equation reads

$$\varphi_B - \varphi_A = \int_{r_A}^{r_B} \frac{b}{r} \frac{\sqrt{\mathcal{A}(r)\mathcal{B}(r)}}{\sqrt{r^2 - b^2\mathcal{A}(r)\mathcal{B}(r)}} dr. \tag{12}$$

Let us denote by $r_c$ the usual Euclidean distance between the center of the massive body and the straight line passing through $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, namely

$$r_c = \frac{r_A r_B}{|\boldsymbol{x}_B - \boldsymbol{x}_A|} |\boldsymbol{n}_A \times \boldsymbol{n}_B|. \tag{13}$$

Our method consists in iteratively solving Eq. (12) for $b$ by assuming that the impact parameter admits an expansion in powers of $m/r_c$ as follows

$$b = r_c \left[ 1 + \sum_{n=1}^{\infty} \left(\frac{m}{r_c}\right)^n q_n \right], \tag{14}$$

where the coefficients $q_n$ are functions of $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ to be calculated.

Substituting for $b$ from Eq. (14) into Eq. (11) shows that each perturbation term in Eq. (5) may be written as

$$\mathcal{T}^{(n)}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{c} \left(\frac{m}{r_c}\right)^n \sum_{s=1-n}^{s(n)} A_{ns}(q_1, \ldots, q_n) \times \int_{r_A}^{r_B} \left(\frac{r}{r_c}\right)^s \frac{r_c^{2n+1}}{(r^2 - r_c^2)^{(2n+1)/2}} dr, \tag{15}$$

where $s(1) = 2, s(n) = 2n - 1$ for $n \geq 2$ and the quantities $A_{ns}(q_1, \ldots, q_n)$ are polynomials in $q_1, \ldots, q_n$. Each integral in Eq. (15) is easy to calculate. Equation (12) enables to determine each coefficient $q_i$. Indeed, inserting Eq. (14) into Eq. (10) yields the expansion

$$\frac{d\varphi}{dr} = \frac{r_c}{r} \frac{1}{\sqrt{r^2 - r_c^2}} + \frac{1}{r_c} \sum_{n=1}^{\infty} \left(\frac{m}{r_c}\right)^n \times \sum_{s=1-n}^{2n-1} B_{ns}(q_1, \ldots, q_n) \left(\frac{r}{r_c}\right)^s \frac{r_c^{2n+1}}{(r^2 - r_c^2)^{(2n+1)/2}}, \tag{16}$$

where the quantities $B_{ns}(q_1, \ldots, q_n)$ are also polynomials in $q_1, \ldots, q_n$. In view of the fact that

$$\int_{r_A}^{r_B} \frac{r_c}{r} \frac{dr}{\sqrt{r^2 - r_c^2}} = \varphi_B - \varphi_A,$$

it results from Eq. (16) that Eq. (12) is equivalent to the infinite set of equations

$$\sum_{s=1-n}^{2n-1} B_{ns}(q_1, \ldots, q_n) \int_{r_A}^{r_B} \left(\frac{r}{r_c}\right)^s \frac{r_c^{2n+1} dr}{(r^2 - r_c^2)^{(2n+1)/2}} = 0, \tag{17}$$

where $n = 1, 2, \ldots$ The coefficients $B_{ns}(q_1, \ldots, q_n)$ are linear in $q_n$. So, it will be easy to solve Eq. (17) for $q_1$ when $n = 1$. Knowing $q_1$, $q_2$ will be then determined by Eq. (17) written for $n = 2$, and so on. Thus the whole sequence of the $q_n$ may be iteratively calculated.

## 4. The third-order terms

Equations (6) and (7) are easily recovered by this method. The determination of the 3rd-order term is scarcely any more complicated. We find

$$\mathcal{T}^{(3)}(\boldsymbol{x}_A, \boldsymbol{x}_B) = -\frac{\gamma+1}{c} \frac{m^3}{r_A r_B} \left(\frac{1}{r_A} + \frac{1}{r_B}\right) \times \frac{|\boldsymbol{x}_B - \boldsymbol{x}_A|}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B} \left[\frac{\kappa \arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|} - \frac{(\gamma+1)^2}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B} + 4(\beta - 1) - \frac{8\beta\gamma + 6\epsilon + 3\beta_3 + \gamma_3}{4(\gamma+1)}\right]. \tag{18}$$

In the Schwarzschild space-time $\kappa = 15/4$ and $(8\beta\gamma + 6\epsilon + 3\beta_3 + \gamma_3)/4(\gamma + 1) = 9/4$.

## 5. Concluding remarks

The calculations required by the method of constrained integration can be performed with any symbolic computation program. It is worthy of note that recovering the well-known expressions of $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(2)}$ constitutes a nice test of reliability for the new procedure.

## References

Ashby, N., & Bertotti, B. 2010, Class. Quantum Grav., 27, 145013
Chandrasekhar, S. 1983, The Mathematical Theory of Black Holes, Oxford University Press, New York
Keeton, C. R., & Petters, A. O. 2005, Phys. Rev. D, 72, 104006
Klioner, S. A., & Zschocke, S. 2010, Class. Quantum Grav., 27, 075015
Le Poncin-Lafitte, C., Linet, B., & Teyssandier, P. 2004, Class. Quantum Grav., 21, 4463
Sarmiento, A. F. 1982, Gen. Rel. Grav., 14, 793
Teyssandier, P., & Le Poncin-Lafitte, C. 2008, Class. Quantum Grav., 25, 145020
Teyssandier, P. 2010, ArXiv,1012.5402v3