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Nonlinear force-free field extrapolation: numerical methods and applications

S. Régnier

School of Mathematics and Statistics, University of St Andrews, St Andrews, Fife, KY16 9SS, United Kingdom
e-mail: stephane@mcs.st-andrews.ac.uk

Abstract. To model 3D coronal magnetic fields, we use different assumptions: the potential field, the linear force-free field and the nonlinear force-free field. The latter assumption requires the knowledge of the three components of the magnetic field at the bottom boundary (photosphere or chromosphere). The recent development of new spectro-polarimetric instruments allows a more accurate and more systematic measurement of the three components of the magnetic field. Before we can make use of the full potential of these instruments, we need to review our knowledge on nonlinear force-free modelling and the solar physics that can be done with those computations. We will summarise the different numerical methods used to determine the coronal magnetic field, and we will review the physical processes and properties derived from the computed magnetic configurations (e.g., magnetic reconnection, energy storage, source of energetic particles).

Key words. Sun: magnetic field – Sun: atmosphere – Sun: corona – Sun: flares – Sun: filaments – Sun: observations

1. Introduction

It is currently difficult to obtain 3D information from measurement of the magnetic field at the top of chromosphere or in the corona. To our knowledge, only measures of the Hanle effect in prominences (Leroy et al. 1984) and of radio brightness maps associated with optically thick harmonics of gyroemission frequencies (White & Kundu 1997) have been capable of estimating the magnetic field in the corona. Other attempts using infrared lines are on the way. Therefore a different approach has to be taken in order to determine the three dimensional structure of the coronal magnetic field. One common approach is to extrapolate the magnetic field from photospheric measurements assuming that the observed magnetic field is in an equilibrium state between the gas pressure, the gravity and the magnetic force:

$$-\nabla p + \rho \boldsymbol{g} + \boldsymbol{j} \wedge \boldsymbol{B} = \boldsymbol{0} \quad (1)$$

where p is the gas pressure, ρ the density of the plasma, $\boldsymbol{g}$ the gravity field in the corona, $\boldsymbol{j}$ the electric current density and $\boldsymbol{B}$ the magnetic field. It has been shown (Woltjer 1958; Gold & Hoyle 1960) that the pressure and gravity forces can be neglected in large parts of the corona: the gravity scale height is large compared to the variation of the magnetic field and the gas pressure, and the coronal plasma β (ratio of the gas pressure and of the magnetic pressure) is, on average, less than 1 from

Send offprint requests to: S. Régnier

the top of the chromosphere to about 2.5 solar radii. Neglecting the gas pressure and the gravity leads to the so-called force-free fields for which only the magnetic force is taken into account to determine the magnetic field configuration in the corona. The force-free assumption, $\boldsymbol{j} \wedge \boldsymbol{B} = \boldsymbol{0}$, leads to three different types of magnetic field:

- the potential field corresponds to a minimum energy state. The extrapolation methods require only the vertical (radial) magnetic field component at the bottom boundary layer (e.g. Schmidt 1964; Semel & Rayrole 1968). This is a well-posed boundary value problem given a unique solution for prescribed boundary conditions.
- the linear force-free field given by

  $$\boldsymbol{\nabla} \wedge \boldsymbol{B} = \alpha \boldsymbol{B} \tag{2}$$

  where α is a constant in the volume. The boundary conditions are the vertical component of $\boldsymbol{B}$ and a guess for the α value. This is an ill-posed boundary value problem (Chiu & Hilton 1977). Several numerical schemes based on Fourier transform or Green's function are commonly used (Nakagawa & Raadu 1972; Chiu & Hilton 1977; Alissandrakis 1981; Semel 1988; Gary 1989). The value of α has to be guessed, to be adjusted to coronal observations (see e.g., Wiegelmann & Neukirch 2002; Carcedo et al. 2003), or to be derived from vector magnetograms (e.g. Leka & Skumanich 1999; Leka et al. 2005).
- the nonlinear force-free (*nlff*) field satisfies the following equations:

  $$\boldsymbol{\nabla} \wedge \boldsymbol{B} = \alpha \boldsymbol{B}, \tag{3}$$

  $$\boldsymbol{B} \cdot \boldsymbol{\nabla} \alpha = 0 \tag{4}$$

  where α is function of space, and from Eqn. (4), α is a constant along a given field line. We will describe the different numerical techniques used to derive the *nlff* field in the section below.

It is important to notice that the non-constant distribution of α at the bottom boundary (photosphere or chromosphere) can be derived from the measurements of the full magnetic field vector at a particular layer:

$$\alpha = \frac{1}{B_z} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right). \tag{5}$$

2. Numerical methods

2.1. Mathematical problem to solve

The first question to ask ourselves when computing a *nlff* magnetic field is what problem do we want to solve. Two different approaches are considered:

(i) finding the nonlinear force-free field associated with boundary conditions corresponding to a mathematically well-posed problem;
(ii) finding the equilibrium the closest to a force-free field and matching the boundary conditions. This corresponds to an ill-posed problem.

The item (i) was first detailed by Sakurai (1981) and later developed by Aly (1989) and Amari et al. (1999). To find the nonlinear force-free field satisfying Eqns. (3-4), the suitable boundary conditions are the distribution of the vertical component of the magnetic field everywhere on the boundaries, and the distribution of α (or J_z) in only one polarity (positive or negative). With those boundary conditions, the problem is a mathematically well-posed problem: in the Hadamard sense, that means that we can in principle prove the existence, uniqueness and stability of the solution. Nevertheless the proofs are not easy to find. Bineau (1972) has proved the existence and uniqueness of the solution in simple connected domains for "small" α values. The proof of existence was extended by Boulmezaoud & Amari (2000) to multiple connected domains (with a complex topology of the magnetic field).

The item (ii) solves the *nlff* equations in order to find the closest equilibrium matching the boundary conditions given by all three components of the magnetic field everywhere on the boundaries. This is a mathematically il-posed problem (like the linear force-free problem) for which there is no proof of existence, uniqueness and stability of solutions. The results of the computation can be far from force-free. The first example of the *nlff* as an ill-posed problem was defined by Wu et al. (1990): using the three components of the magnetic field at the photospheric level, the authors extrapolated the magnetic field for different height using the vertical integration method. As shown by Démoulin et al. (1992), it appears that the solution becomes unstable with height and no smoothing functions were found in order to obtain a decaying field.

2.2. Method overview

In this section, we will describe the different numerical schemes that are used to derived the properties of the coronal magnetic field assuming a nonlinear force-free field equilibrium. The methods are: the Grad & Rubin methods (Grad & Rubin 1958), the optimization schemes, the MHD evolutionary techniques, the boundary element method and the finite element method. For each method, different implementation and numerical schemes are considered.

Grad & Rubin methods.

Currently four Grad & Rubin numerical schemes are used to derive the *nlff* field: Amari et al. (1999), Wheatland (2004, 2006), Inhester & Wiegelmann (2006) and Amari et al. (2006). In Table 1, the boundary conditions for these methods are given: the vertical component of the magnetic field everywhere and the distribution of α in one chosen polarity.

The basic idea of the Grad & Rubin scheme is to decompose the system of Eqns.(3–4) into two different systems: a hyperbolic part corresponding to the transport of α along field lines, and an elliptic part updating the magnetic field configuration. The methods mentioned below differ in the way of solving the different systems.

The Grad & Rubin method as implemented by Wheatland (2004, 2006) is similar to that of Sakurai (1981) where the current distribution is modeled in terms of cylindrical current elements between nodal points on a small number of calculated field lines. The magnetic force is calculated at each nodal point due to all current elements using an exact integral solution of the Ampère's law. In Wheatland (2004), the magnetic field due to the currents is calculated at each gridpoints instead of only at nodal points.

Amari et al. (1999) solve the Grad & Rubin problem in two different iterative steps: the hyperbolic part is solved for a fraction of the current density and then the field updated by relaxation. The current density is progressively injected in the magnetic configuration. The numerical scheme is written in terms of the vector potential $\boldsymbol{A}$ associated with $\boldsymbol{B}$. This ensures that the solenoidal constraint is satisfied. Amari et al. (2006) have improved the numerical scheme in many ways, e.g. by injecting the current density only in one step, by ensuring the $\boldsymbol{\nabla}\cdot\boldsymbol{j} = \boldsymbol{0}$ at a high level of accuracy, or by improving the determination of the boundary values of the transverse component of the vector potential (see Amari et al. 2006)

The method implemented by Inhester & Wiegelmann (2006) first propagates the α value along field lines and then updates the magnetic field using a residual vector potential and also ensuring the condition $\boldsymbol{\nabla}\cdot\boldsymbol{j} = \boldsymbol{0}$.

Optimization schemes.

The optimization scheme was first implemented by Wheatland et al. (2000), and later developed by Wiegelmann (2004). Note that another optimization scheme has been implemented by McTiernan (see Schrijver et al. 2006).

The basic principle is to minimize a functional (L) containing the normalized magnetic force and the solenoidal condition:

$$L = \frac{1}{V}\int_V \left(\frac{|(\boldsymbol{\nabla}\wedge\boldsymbol{B})\wedge\boldsymbol{B}|^2}{B^2} + |\boldsymbol{\nabla}\cdot\boldsymbol{B}|^2\right) dV \quad (6)$$

By introducing an artificial time parameter, the minimisation process can be written as:

$$\frac{dL}{dt} = -2\int_V \kappa F^2 \, dV \quad (7)$$

where $\boldsymbol{F}$ is a function of $\boldsymbol{B}$ and its second-order derivatives. The magnetic field is then updated by $\boldsymbol{B}^{(n+1)} = \boldsymbol{B}^{(n)} + \boldsymbol{F}^{(n)}\Delta t$.

Table 1. Boundary conditions and initial states used to derive the *nlff* field in the corona from photospheric or chromospheric boundary conditions for several numerical schemes described in the text.

Method	bottom boundary conditions	boundaries on other sides	initial state	preprocess and comments
Grad & Rubin				
by Amari et al. (99)	B_z and $\alpha^{\pm}$	closed	potential	smooth α
by Sakurai	”	”	”	no
by Wheatland	”	no	”	no
by Inhester	”	B_n unchanged	”	no
by Amari et al. (06)	”	B_n	”	no
Optimization				
by Wheatland et al.	B_x, B_y, B_z	B_x, B_y, B_z	potential	no
by Wiegelmann	”	”	”	minimizing forces and torques
Evolutionary technique				
by Mikic et al.	B_z, J_z	closed	potential	external circuit
Boundary elements				
by Yan and Sakurai	B_x, B_y, B_z	closed	no	no
Finite elements				
by Amari et al. (06)	B_z, $\alpha^{\pm}$	B_n, $\alpha^{\pm}$	potential	smooth α

The optimization code has been coupled to a data preprocessing technique (Wiegelmann et al. 2006) which minimizes the forces and torques of the magnetic field in order to get boundary data resembling a force-free field (Aly 1989).

MHD evolutionary techniques.

Different MHD-based techniques have been developed: the evolutionary technique (see e.g., Jiao et al. 1997), the stress-and-relax method (Roumeliotis 1996), and the magnetofrictional method (Yang et al. 1986).

The MHD evolutionary method as implemented by McClymont & Mikic (1994) follows the time-dependent evolution of the resistive, viscous, MHD equations using changing boundary conditions. An incompressible two-dimensional flow is imposed on the boundary in order to inject the observed current density (due to transverse field) in the magnetic configuration. The coronal resistivity is needed to allow the connectivity of field lines to evolve in time (magnetic reconnection).

The stress-and relax method (Roumeliotis 1996) is very similar to the MHD evolutionary technique solving similar MHD equations. But the resistive relaxation is driven by the transverse components of the magnetic field and also includes the uncertainty of the magnetic field measurements.

The magnetofrictional method (Yang et al. 1986) uses a dissipative relaxation to drive the MHD equations towards an equilibrium. The boundary conditions are injected by a series a stress-and-relax procedures. This method has been recently implemented by Valori et al. (2005) with a zero plasma β which results in a final state close to a force-free state.

Boundary integral method.

Yan & Sakurai (1997, 2000) have proposed an integral equation representation of the *nlff* field model considering the half-space above the lower boundary with vanishing field at infinity. Following Green's second identity, the solution at a given point i is given by:

$$c_i B_i = \oint_S \left(\boldsymbol{Y} \frac{\partial \boldsymbol{B}}{\partial n} - \frac{\partial \boldsymbol{Y}}{\partial n} \boldsymbol{B} \right) dS \qquad (8)$$

where c is a constant (1 or 0.5) and $\boldsymbol{Y}$ is an auxiliary function.

An iterative scheme has been developed (Yan & Sakurai 2000; Li et al. 2004) to compute the *nlff* field at given point in the coronal volume from the boundary conditions given by the 3 components of the magnetic field. Note that, as implemented in Yan & Sakurai (2000), a volume integral is needed to determine the auxiliary function at a given point. Yan & Li (2006) have recently implemented a new version of the boundary integral method without the volume integral.

Finite element method.
The hyperbolic-elliptic system of the *nlff* field as described in the Grad & Rubin method section can be discretized using the finite element technique. The implementation of the finite elements has been done by Amari et al. (2006). It is important to note that the elliptic part is solved with a discretization on non divergence-free finite elements. For the hyperbolic part, a linear system is solved, instead of propagating α along a field line using a field line tracing technique Amari et al. (1999). The boundary conditions are similar to the Grad & Rubin method and then also corresponds to a well-posed problem (see Table 1).

2.3. Comparison of methods

The different methods have been tested for convergence and accuracy with semi-analytical solutions. The tests have been performed with the Low & Lou (1990) solutions which provide a large set of *nlff* solutions. The comparison has been done by Schrijver et al. (2006), Amari et al. (2006) and Inhester & Wiegelmann (2006). The authors have used different figures of merit to compare the different solutions. In Schrijver et al. (2006), six different methods are compared. The boundary conditions, the initial state (potential field) and the grid are different from one model to the other. The *nlff* computations agree in the strong field regions, discrepancies arise in the weak field regions. In Amari et al. (2006), the authors compare the Grad & Rubin method and the finite elements method. In Inhester & Wiegelmann (2006), the authors compare the Grad & Rubin method and the optimization scheme. The authors have implemented both methods on the same grid for a better comparison. Again the computed fields agree in strong field region, even if each method has its own convergence properties.

3. Applications to solar active regions

The methods described above can be in practice applied to the 3D magnetic field in the corona under the *nlff* assumption using observed vector magnetic field measurements (photospheric and chromospheric). So far, few of the numerical schemes have actually been applied to solar vector magnetograms and have produced significant results for the understanding of coronal magnetic fields and the associated physical processes: the evolutionary technique (Mikic & McClymont 1994; Lee et al. 1998), the boundary integral method (Wang et al. 2000; Yan et al. 2001a,b; Liu et al. 2002), the Grad & Rubin method (Régnier et al. 2002; Bleybel et al. 2002; Régnier & Amari 2004; Régnier et al. 2005a; Régnier & Canfield 2006) and the optimization method (Wiegelmann et al. 2005a,b, 2006).

3.1. Magnetic structures

Once the 3D magnetic configuration is reconstructed, we first have to check if characteristic magnetic structures can be obtained in comparison with observations. One first can notice that the *nlff* field give better qualitative results than the potential and the linear force-free fields in most cases as it has be shown by Mikic & McClymont (1994); Bleybel et al. (2002); Wiegelmann et al. (2005b). The field lines obtained from the *nlff* methods are well correlated to the field lines (heated or over-dense loops) observed in EUV by SOHO/EIT or TRACE at temperature of about 1 million degrees (see e.g. Wiegelmann et al. 2005b; Régnier & Canfield 2006).

By introducing a non-constant distribution of α, we can reconstruct magnetic structures within a large range of twist and shear. This is important in order to reconstruct structures such as filaments and sigmoids. In Yan et al. (2001a) and Régnier & Amari (2004), the authors have proved the existence of highly twisted flux tubes in active regions describing filaments and sigmoids:

- Yan et al. (2001a) have shown that a highly twisted flux tube is associated with the Hα filament involved in the Bastille day flare in 2000; the rope structure has been found with a number of turns more than 3;
- Régnier & Amari (2004) have found that, in a decaying active region with high current density, twisted flux bundles with different number of turns can be reconstructed and compared to the soft X-rays sigmoid and the Hα filament observed in the active region.

In both cases, the authors have deduced that the flux rope instability is responsible for the eruptive event associated with those particular active regions. In Yan et al. (2001a), the kink instability of the flux rope rising in the corona has triggered the X5 flare and the energetic events associated with the Bastille Day flare. In Régnier & Amari (2004), the kink instability of the twist flux bundle with a number of turns of about 1.2 corresponds to the onset of a slow CME without detectable flare.

3.2. Chromospheric and coronal signatures

In addition to determining the structures involved in eruption processes, we can also correlate the magnetic structures and different observational signatures: chromospheric and EUV intensity enhancements, chromospheric blueshifted events and radio sources.

Mikic & McClymont (1994) have matched the Hα signatures of a flare with *nlff* field lines in both polarities showing that the coronal enhancement due to a flare is related to the chromospheric features by the same set of field lines. Using Hα images, Wang et al. (2001) have matched the plage regions and the chromospheric fibrils with small-scale field lines. From the measurement of Doppler shifts in the Hα line, Régnier & Canfield (2006) have shown that the occurrence of a series of flares is related to blueshifted events which are signatures of reconnected field lines.

Lee et al. (1998) have compared the radio observations and *nlff* field lines. The authors have combined observations at gyroresonance frequencies (optically thick) and coronal heating of reconstructed field lines in order to determine the radio brightness temperature maps. The authors have shown that the connectivity of *nlff* field lines agrees well with the connectivity inferred from the radio observations. This method also allows to extract some geometrical properties of the radio sources.

3.3. Time evolution

The results mentioned above have been obtained for a snapshot of an active region at a given time. As suggested by Antiochos (1987), it should be possible to reconstruct the time evolution of an active region by successive equilibriums if the evolution is slow enough to allow for the relaxation of the magnetic configuration between two successive times (Δt greater than several Alfvén times). This method of successive equilibria has recently been successfully applied by Régnier & Canfield (2006) for a time range less than 5 hours. Using a time series of vector magnetograms with a Δt of about 20 min, Régnier & Canfield (2006) have treated the equilibria independently of each other. They have first shown the smooth evolution of the field configuration in time. In other words, there is no bifurcation of the magnetic configuration from one equilibrium to the following equilibrium which can result from slightly different boundary conditions or different level of noise. During the time evolution of the active region, the authors have noticed that small changes occurred in the magnetic configurations. And by relating those changes to chromospheric and coronal observations, they have shown that these changes are signatures of magnetic reconnection processes of field lines from one connectivity domain to another. The authors argue that the flaring activity in this particular active region during this time period is due to reconnection of magnetic field lines at the boundaries of connectivity domains (or separatrix surfaces) caused by photospheric motions (sunspot rotation and apparent motion of an emerging polarity).

This method has so far been applied to only one active region. It requires data of high quality especially in terms of the noise of the transverse components. Nevertheless it is promising for the future use of vector magnetograms and our understanding of solar eruptions.

3.4. Magnetic energy budget

An other important problem that can be tackled with *nlff* reconstruction techniques is to estimate of magnetic energy budget in a configuration and its evolution in time. The magnetic energy is given by:

$$E_m = \int_V \frac{B^2}{2\mu_0}\, dV \tag{9}$$

Most of the papers mentioned above has measured the magnetic energy of *nlff* configurations, giving a range of energy between 10^{31} and 10^{34} erg depending on the photospheric magnetic flux and the volume used to model the 3D field. Nevertheless the most important quantity is the free magnetic energy budget deduced from the *nlff* and the potential fields given by

$$\Delta E_m = E_m^{nlff} - E_m^{pot} \tag{10}$$

For the different active regions, the free magnetic energy budget is often between 10^{30} and 10^{32} erg corresponding to a percentage of free energy in a magnetic configuration between 5% and 45% with respect to the *nlff* magnetic energy. The estimated amounts have been shown to always be sufficient to trigger the flare associated with the studied active region.

Bleybel et al. (2002) and Liu et al. (2002) have used two snapshots to determine the energy budget before and after a flare. The energy budget has been shown to decrease after the flare as expected. It should be noted that this result cannot be generalized: if the time between the two snapshots is too long (several hours), the changes in magnetic energy will not necessarily be related to the flare but can also be due to a change of energy caused by photospheric motions. As an example, Régnier et al. (2005b) have shown that, for a particular active region, the energy injected by the emerging magnetic flux was more important than the energy released during a flare.

With the successive equilibrium method, the evolution of the magnetic energy over a short period oftime can also be studied. In particular, Régnier & Canfield (2006) have shown that there is an increase of the magnetic energy before the flare and a decrease of magnetic energy of about the same amount after the flare. This work was done for C-class flares. The rate of change of the magnetic energy was estimated to be 10^{28} erg$\cdot$s^{-1}.

3.5. Magnetic helicity

Magnetic helicity is important to understand how magnetic field lines are twisted or intertwined. The magnetic helicity is defined by:

$$H_m = \int_V \boldsymbol{A} \cdot \boldsymbol{B}\, dV \tag{11}$$

where $\boldsymbol{A}$ is the vector potential associated with $\boldsymbol{B}$ ($\boldsymbol{B} = \boldsymbol{\nabla} \wedge \boldsymbol{A}$). As the previous definition of the magnetic helicity is not gauge invariant, unless the volume V is bounded by flux surfaces, it is better to define a relative magnetic helicity (Berger & Field 1984; Finn & Antonsen 1985) by:

$$\Delta H_m = \int_V (\boldsymbol{A} - \boldsymbol{A}_{pot}) \cdot (\boldsymbol{B} + \boldsymbol{B}_{pot})\, dV. \tag{12}$$

To compute the magnetic helicity, we then have to calculate the vector potential. So far only the Grad & Rubin method implemented by Amari et al. (1999) uses the vector potential to compute the 3D magnetic field. Bleybel et al. (2002) and Régnier & Amari (2004) have measured the relative magnetic helicity of active regions. Typically, the values of magnetic helicity found in active regions range from 10^{41} to 10^{43} $G^2 \cdot cm^4$. It is important to notice that the magnetic helicity of an active region is not constant in time due to the injection or cancellation of magnetic helicity in the considered finite coronal domain. For an active region following the Joy's law, the sign of the magnetic helicity is usually consistent with the chirality rules (Pevtsov et al. 1995; Longcope et al. 1998).

Due to the boundary conditions used by the Grad & Rubin method (Amari et al. 1999), the magnetic field in the coronal volume can be decomposed into the sum of a closed field and a reference field as described by Berger (1999). This decomposition allows us to separate the relative magnetic helicity into two terms: the self helicity and the mutual helicity. The self helicity estimates the amount of twist of confined flux bundles, and the mutual helicity evaluates the crossing of field lines as well as the large scale twist of the flux tube (see Régnier et al. 2005a). Régnier & Canfield (2006) have shown that the eruptive phenomena associated with a particular active region are followed by an injection of negative helicity or a cancellation of positive helicity.

4. Conclusions

In the near future, a lot of vector magnetic field measurements will be available from ground-based observatories (THEMIS, SOLIS, GREGOR, ATST, ...) and from space missions (Hinode, SDO, Solar Orbiter). One challenge that we will face is how to handle such large amounts of data in order to reliably and routinely determine the 3D magnetic field in the corona from photospheric and chromospheric measurements. The first steps are (i) to summarize our current knowledge on field extrapolations, (ii) to understand which mathematical problem we want to solve (see Section 2.1), (iii) to find a compromise between speed and accuracy of the reconstruction methods.

The *nlff* reconstruction methods are currently the most advanced numerical techniques to find the 3D structure of the coronal magnetic field. In the recent year, we have made progress in understanding the weaknesses and the strengths of each method. Without ruling out any of those methods, it is important to define precisely the physical problem to address in order to chose the most suitable method. As shown by Schrijver et al. (2006), more tests are needed to really understand the differences between the methods and their relative advantages and disadvantages.

We have recently seen a growing interest for going beyond the *nlff* modeling of the coronal field by introducing the extrapolation of non force-free fields, e. g. magnetohydrostatic fields (Wiegelmann & Neukirch 2006).

It is important to note that those computations rely on the accurate measurements of the vector magnetic field at a given layer of the photosphere or the chromosphere. The recent progress in inverting the Stokes parameters and in solving the 180^o ambiguity on the transverse components are essential for reconstruction techniques.

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